HW5 ※12-4: 4, 8, 10, 18, 24, 28, 34, 36, 43,

※12-5: 6, 12, 20, 28, 30, 36, 44, 45,

§12-4

4. $\sum \left(\frac{k}{2k+1}\right)^k$: converges

Let $a_k = \left(\frac{k}{2k+1}\right)^k > 0$, as $k \geq 1$

$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} < 1$, by Root test, converges.

8. $\sum \frac{1}{(\ln k)^k}$: converges.

Let $a_k = \frac{1}{(\ln k)^k} > 0$, as $k \geq 1$

$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{1}{\ln k} = 0 < 1$, by Root test, converges.

10. $\sum \frac{1}{(\ln k)^{10}}$: diverges.

Ratio test fails.

Let $a_k = \frac{1}{(\ln k)^{10}} > 0$, as $k \geq 1$

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{(\ln(k+1))^{10}}}{\frac{1}{(\ln k)^{10}}} = \lim_{k\to\infty} \left(\frac{\ln k}{\ln(k+1)}\right)^{10} = \left(\lim_{k\to\infty} \underbrace{\frac{\ln k}{\ln(k+1)}}_{\left(\frac{\infty}{\infty}\right)}\right)^{10} = \left(\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{k+1}}\right)^{10} = 1$

Let $f(x) = x^{\frac{1}{10}} - \ln x$

$$f'(x) = \frac{1}{10}\cdot x^{\frac{-9}{10}} - \frac{1}{x} = \frac{1}{10}\cdot\frac{1}{x}\left(x^{\frac{1}{10}} - 10\right) \geq 0 \iff x^{\frac{1}{10}} - 10 \geq 0,\ x \geq 10^{10}$$

as $k \geq [10^{10}]+1$ (integer), we have $f(k) = k^{\frac{1}{10}} - \ln k \geq 0 \Rightarrow k^{\frac{1}{10}} \geq \ln k$, as $k \geq [10^{10}]+1$.

$$\Rightarrow k \geq (\ln k)^{10} \Rightarrow \frac{1}{(\ln k)^{10}} \geq \frac{1}{k}$$

so, by comparison test, $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^{10}} \geq \sum_{k=[10^{10}]+1}^{\infty} \frac{1}{(\ln k)^{10}} \geq \sum_{k=[10^{10}]+1}^{\infty} \frac{1}{k}$: diverges.

18. $\sum \frac{1}{k}\cdot\left(\frac{1}{\ln k}\right)^{3/2}$: converges.

Let $a_k = \frac{1}{k}\left(\frac{1}{\ln k}\right)^{3/2} > 0$, as $k \geq 2$ ($\because a_1$ is undefined)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{k+1}\cdot\left(\frac{1}{\ln(k+1)}\right)^{3/2}}{\frac{1}{k}\cdot\left(\frac{1}{\ln k}\right)^{3/2}} = 1,$$ Ratio test fails

by integral test :

$$\int_2^{\infty} \frac{1}{x}\cdot\left(\frac{1}{\ln x}\right)^{3/2} dx = \int_{\ln 2}^{\infty} u^{-\frac{3}{2}}\, du = -2u^{-\frac{1}{2}}\Big|_{\ln 2}^{\infty} = \frac{2}{\sqrt{\ln 2}} < \infty$$ $\therefore$ converges

$u = \ln x$

$du = \frac{1}{x}dx$

24.

$$\sum \frac{11}{1+100^{-k}} : \text{diverges.}$$

Let $a_k = \frac{11}{1+100^{-k}} > 0$, as $k \geq 1$

since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{11}{1+100^{-k}} = 11 \neq 0$, so $\sum_{k=1}^{\infty} a_k$: diverges.

28.

$$\sum \frac{k!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)} : \text{converges.}$$

Let $a_k = \frac{k!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)} > 0$, as $k \geq 1$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{\cancel{(k+1)!}^{\,k+1}}{\cancel{1\cdot 3\cdot 5 \cdots (2k-1)}\cdot(2k+1)}}{\frac{\cancel{k!}}{\cancel{1\cdot 3\cdot 5\cdots(2k-1)}}} = \lim_{k\to\infty} \frac{k+1}{2k+1} = \frac{1}{2} < 1$$

by Ratio test, converges.

34. $$\sum \frac{k^k}{(3^k)^2} : \text{diverges.}$$

Let $a_k = \frac{k^k}{(3^k)^2} > 0$, as $k \geq 1$, $\left(\because \lim_{x\to\infty} x^{x} = 1\right)$

since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^k}{(3^k)^2} = \lim_{k\to\infty} \left(\frac{k}{9}\right)^k = \lim_{k\to\infty}\left(\left(\frac{k}{9}\right)^{\frac{k}{9}}\right)^9 = \left(\lim_{k\to\infty}\left(\frac{k}{9}\right)^{\frac{k}{9}}\right)^9 = 1 \neq 0$

So, $\sum_{k=1}^{\infty} a_k$: diverges.

36.

$\sum (\sqrt{k} - \sqrt{k-1})^k$: converges.

Let $a_k = (\sqrt{k} - \sqrt{k-1})^k > 0$, as $k \geq 1$

$$\lim_{k\to\infty} (a_k)^{\frac{1}{k}} = \lim_{k\to\infty} (\sqrt{k} - \sqrt{k-1}) = \lim_{k\to\infty} \frac{1}{\sqrt{k} + \sqrt{k-1}} = 0 < 1$$

by Root test, converges

43. $r > 0$, show that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{r^k}{k!} = 0$ ☆ (use "series converges" to discuss "the limit of sequence".)

<pf>

Let $a_k = \frac{r^k}{k!} > 0$, as $k \geq 1$, $r > 0$,

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{r^{k+1}}{(k+1)!}}{\frac{r^k}{k!}} = \lim_{k\to\infty} \frac{r}{k+1} = 0 < 1$$

, by Ratio test,

$\Rightarrow \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{r^k}{k!}$ converges

by theorem 12.2.5 (P.583)

then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{r^k}{k!} = 0$ ▨

§12-5

※6.

$\sum (-1)^k \cdot \frac{k}{\ln k}$: diverges

Let $a_k = \frac{k}{\ln k} > 0$, as $k \geq 1$

since $\lim_{k\to\infty} \frac{k}{\ln k} = \lim_{k\to\infty} \frac{1}{\frac{1}{k}} = \lim_{k\to\infty} k = \infty \neq 0$

So, $\lim_{k\to\infty} (-1)^k \cdot \frac{k}{\ln k} \neq 0$, then $\sum (-1)^k \cdot \frac{k}{\ln k}$: diverges.

※12.

$\sum \sin\left(\frac{k\pi}{4}\right)$: diverges

Let $a_k = \sin\left(\frac{k\pi}{4}\right) \Rightarrow a_1 = \frac{1}{\sqrt{2}}$, $a_2 = 1$, $a_3 = \frac{1}{\sqrt{2}}$, $a_4 = 0$, $a_5 = \frac{-1}{\sqrt{2}}$, $a_6 = -1$, $a_7 = \frac{-1}{\sqrt{2}}$,

$a_8 = 0$, ……

$\Rightarrow \lim_{k\to\infty} a_k =$ diverges.

$\Rightarrow \sum \sin\left(\frac{k\pi}{4}\right)$: diverges.

20.

$$\sum (-1)^k \cdot \frac{k+2}{k^2+k}$$

(a)

$$\sum \left| (-1)^k \cdot \frac{k+2}{k^2+k} \right| = \sum \frac{k+2}{k^2+k}$$

$\because \frac{k+2}{k^2+k} \geq \frac{k+2}{2k^2} \geq \frac{k}{2k^2} = \frac{1}{2k} \Rightarrow \sum \frac{k+2}{k^2+k} \geq \sum \frac{1}{2k}$, by comparison test,

since $\sum \frac{1}{2k}$: diverges, so $\sum \frac{k+2}{k^2+k}$: diverges, thus, $\sum \left| (-1)^k \cdot \frac{k+2}{k^2+k} \right| = \sum \frac{k+2}{k^2+k}$: diverges

Not absolute convergence

(b)

$\sum (-1)^k \cdot a_k = \sum (-1)^k \cdot \frac{k+2}{k^2+k}$, $a_k = \frac{k+2}{k^2+k} \geq 0$

$$\frac{a_{k+1}}{a_k} = \frac{\frac{k+3}{(k+1)^2+k+1}}{\frac{k+2}{k^2+k}} = \frac{\frac{k+3}{k^2+3k+2}}{\frac{k+2}{k^2+k}} = \frac{(k+3)(k^2+k)}{(k+2)(k^2+3k+2)} = \frac{k^3+4k^2+3k}{k^3+5k^2+8k+4} \leq 1 \Rightarrow a_{k+1} \leq a_k$$

(decreasing)

$\lim_{k\to\infty} a_k = \frac{k+1}{k^2+k} = 0 \Leftrightarrow \sum (-1)^k \cdot a_k$: converges. (alternating series), so conditional convergence.

18. $\sum \frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}}$: absolute converges.

$a_k = \frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}} \Rightarrow a_1 = \frac{1}{1}$, $a_2 = \frac{0}{2\sqrt{2}}$, $a_3 = \frac{-1}{3\sqrt{3}}$, $a_4 = \frac{0}{4\sqrt{4}}$,

$a_5 = \frac{1}{5\sqrt{5}}$, $a_6 = \frac{0}{6\sqrt{6}}$, $a_7 = \frac{-1}{7\sqrt{7}}$, $a_8 = \frac{0}{8\sqrt{8}}$, ...

$\sum \left| \frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}} \right| = \sum \frac{1}{k\sqrt{k}}$, by integral test,

$\int_1^\infty \frac{1}{x\sqrt{x}} dx = -2 \cdot x^{-\frac{1}{2}} \Big|_1^\infty = 2 < \infty \Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$: converges

So, absolute convergence.

§12-5

※ 30.

$$\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\cdots+\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}+\cdots$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}\right)$$

suppose $\sum_{k=0}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}\right)$ converges. — ①

Now, $\sum_{k=1}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)=\sum_{k=1}^{\infty}\frac{1}{(3k+2)(3k+3)}\leq\sum_{k=1}^{\infty}\frac{1}{k^2}$

since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, by comparision test, so $\sum_{k=1}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)$: converges — ②

$$\sum_{k=1}^{\infty}\frac{1}{3k+4}=\sum_{k=1}^{\infty}\left[\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)-\left(\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}\right)\right]$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)-\sum_{k=1}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}\right)\quad(\because ①, ②)$$

converges.

but, $\sum_{k=1}^{\infty}\frac{1}{3k+4}$: diverges (contradiction)

so, $\sum_{k=0}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}\right)$: diverges.

##

36. the series $\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{10^k} = L$, $S_n = \sum_{k=0}^{n} (-1)^k \cdot \frac{1}{10^k}$

use (12.5.4) , $a_n = \frac{1}{10^n}$

$$|S_n - L| < a_{n+1}$$

(a)

$$|S_n - L| < a_{n+1} = \frac{1}{10^{n+1}} < 10^{-3} = \frac{1}{1000} \Rightarrow n \geq 3$$

(b)

$$|S_n - L| < a_{n+1} = \frac{1}{10^{n+1}} < 10^{-4} = \frac{1}{10000} \Rightarrow n \geq 4$$

44. if Σa_k is absolutely convergent and $|b_k| \leq |a_k|$, for all k, then Σb_k is absolutely convergent.

<pf>

since Σa_k is absolutely convergent, then $\Sigma |a_k|$ converges.

Since $|b_k| \leq |a_k|$, by comparison test, then $\Sigma |b_k|$ converges.

Then, Σb_k is absolutely convergent.

$\blacksquare$

45. (a) Show that if $\sum a_k$ is absolutely convergent, then $\sum a_k^2$ is convergent

(b) Show by means of an example that the converse of the result in part (a) is false.

<pf of (a)>

since $\sum a_k$ is absolutely convergent, then $\sum |a_k|$ convergent.

$\sum a_k^2 = \sum |a_k|^2$ converges. (by §12-3 ※49)

<Example of (b)>

$a_k = (-1)^k \cdot \frac{1}{k}$, $a_k^2 = \frac{1}{k^2}$

$\sum a_k^2 = \sum \frac{1}{k^2}$ converges. , but $\sum a_k = \sum (-1)^k \cdot \frac{1}{k}$: is not absolutely convergent.

($\because$ integeral test). (since $\sum \frac{1}{k}$: diverges).

#

→ §12-3 ※49.

※ if $\sum a_k$ converges, then $\sum a_k^2$ converges. (why?)
$a_k \geq 0$.

<pf>

since $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

choose $0<\varepsilon<1$, there is exist $N \in \mathbb{N}$ such that $|a_k - 0| = |a_k| < \varepsilon < 1$, as $k \geq N$

$\Rightarrow 0 \leq a_k < 1$, as $k \geq N$

$\Rightarrow a_k^2 \leq a_k < 1$, as $k \geq N$

$\Rightarrow \sum_{k=N}^{\infty} a_k^2 \leq \sum_{k=N}^{\infty} a_k$, by comparison test, since $\sum_{k=1}^{\infty} a_k$ converges.

$\Rightarrow \sum_{k=N}^{\infty} a_k^2$ converges.

$\Rightarrow \sum_{k=1}^{\infty} a_k^2$ converges.