Hw6: §12-6: 2, 4, 8, 10, 11, 14, 28, 38, 46,

§12-7: 2, 6, 8, 12, 20, 26, 30

§12-6

#2. $f(x)=\sqrt{1+x}$

$f^{(1)}(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{1+x}}$

$f^{(2)}(x)=\frac{-1}{4}\cdot(1+x)^{-\frac{3}{2}}$

$f^{(3)}(x)=\frac{3}{8}(1+x)^{-\frac{5}{2}}$

$f^{(4)}(x)=\frac{-15}{16}(x+1)^{-\frac{7}{2}}$

$$P_4(x)=\sum_{k=0}^{4}\frac{f^{(k)}(0)}{k!}\cdot x^k$$

$$=f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$=\underline{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4}$$

#4. $f(x)=\sec x$, $f^{(1)}(x)=\sec x\cdot\tan x$, $f^{(2)}(x)=\sec x\cdot\tan^2 x+\sec^3 x=\sec x\,(2\tan^2 x+1)$

$$f^{(3)}(x)=\sec x\tan x\,(1+2\tan^2 x)+\sec x\cdot 4\tan x\cdot\sec^2 x=\sec x\,(\tan x+2\tan^3 x+4\tan x+4\tan^3 x)$$

$$=\sec x\,(6\tan^3 x+5\tan x)$$

$$f^{(4)}(x)=\sec x\tan x\,(6\tan^4 x+5\tan x)+\sec x\,(18\tan^2 x\cdot\sec^2 x+5\sec^2 x)$$

$$P_4(x)=f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4$$

$$=1+\frac{0}{1!}x+\frac{1}{2!}x^2+\frac{0}{3!}x^3+\frac{5}{4!}x^4$$

$$=\underline{1+\frac{1}{2}x^2+\frac{5}{24}x^4}$$

8. $f(x) = x\cos x^2$, $f^{(1)}(x) = \cos x^2 - x\cdot \sin x^2 \cdot 2x = \cos x^2 - 2x^2 \cdot \sin x^2$

$f^{(2)}(x) = -\sin x^2 \cdot 2x - 4x\cdot \sin x^2 - 2x^2\cdot \cos x^2 \cdot 2x = -6x\sin x^2 - 4x^3 \cdot \cos x^2$

$f^{(3)}(x) = -6\sin x^2 - 6x\cdot\cos x^2\cdot 2x - 12x^2\cdot\cos x^2 + 4x^3\cdot \sin x^2 \cdot 2x$

$= (-6+8x^4)\cdot \sin x^2 - 24x^2\cdot\cos x^2$

$f^{(4)}(x) = 32x^3\cdot\sin x^2 + (-6+8x^4)\cdot\cos x^2\cdot 2x - 48x\cdot\cos x^2 + 24x^2\cdot\sin x^2\cdot 2x$

$= 80x^3\cdot\sin x^2 + (-12x+16x^5-48x)\cdot\cos x^2$

$f^{(5)}(x) = 240x^2\cdot\sin x^2 + 80x^3\cdot\cos x^2\cdot 2x + (-12+80x^4-48)\cdot\cos x^2 - (16x^5-60x)\cdot\sin x^2\cdot 2x$

$P_5(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$

$= 0 + x + 0 + 0 + 0 + \frac{-60}{5!}x^5$

$= x - \frac{1}{2}x^5$

10. $f(x)=(x+1)^3$, $f^{(1)}(x)=3(x+1)^2$, $f^{(2)}(x)=6(x+1)$, $f^{(3)}(x)=6$

$P_0(x)=f(0)=\underline{1}$

$P_1(x)=f(0)+\frac{f^{(1)}(0)}{1!}x=\underline{1+3x}$

$P_2(x)=f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2=1+3x+\frac{6}{2!}x^2=\underline{1+3x+3x^2}$

$P_3(x)=f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3=\underline{1+3x+3x^2+x^3}$

11. $f(x)=e^{-x}$, $f^{(1)}(x)=-e^{-x}$, $f^{(2)}(x)=e^{-x}$, $f^{(3)}(x)=-e^{-x}$, $f^{(4)}(x)\doteq e^{-x}$

$f(0)=1$ $\quad f^{(1)}(0)=-1$ $\quad f^{(2)}(0)=1$ $\quad f^{(3)}(0)=-1$ $\quad f^{(4)}(0)=1$

$f^{(5)}(x)=-e^{-x}$, $f^{(6)}(x)=e^{-x}$, $f^{(7)}(x)=-e^{-x}$, $f^{(8)}(x)=e^{-x}$ $\cdots\cdots$

$f^{(5)}(0)=-1$ $\quad f^{(6)}(0)=1$ $\quad f^{(7)}(0)=-1$ $\quad f^{(8)}(0)=1$ $\cdots\cdots$

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\frac{1}{5!}x^5+\cdots\cdots$$

$$=\underline{\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k}$$

14、

$f(x) = \ln(1-x)$, $f^{(1)}(x) = \frac{-1}{1-x}$, $f^{(2)}(x) = \frac{-1}{(1-x)^2}$, $f^{(3)}(x) = \frac{0-(-1)\cdot 2(1-x)\cdot(-1)}{(1-x)^4}$

$f(0) = \ln 1 = 0$ $\qquad = \frac{-2}{(1-x)^3}$

$$f^{(4)}(x) = \frac{0-(-2)\cdot 3(1-x)^2\cdot(-1)}{(1-x)^6} = \frac{-2\cdot 3}{(1-x)^4} \cdots\cdots$$

$$\Rightarrow f^{(k)}(x) = \frac{-(k-1)!}{(1-x)^k}, \ k \geq 1 \Rightarrow f^{(k)}(0) = -(k-1)! \ , k \geq 1$$

$$\Rightarrow P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{-(k-1)!}{k!} x^k = \underline{\sum_{k=1}^{n} \frac{-1}{k} \cdot x^k}$$

28、

to estimate the $\underline{\underline{\ln 1.2}}$

use (12.6.8)

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\cdots$$

$$x = 0.2 \ , \ \ln 1.2 = 0.2 - \frac{1}{2}\times 0.2^2 + \frac{1}{3}\times 0.2^3 - \frac{1}{4}\times 0.2^4 + \cdots\cdots$$

$$= 0.2 - 0.02 + \underline{0.0026667} - 0.0004 + \cdots\cdots$$

$$= 0.182667 - 0.0004 + \cdots\cdots$$

$$= 0.182267 \cdots\cdots$$

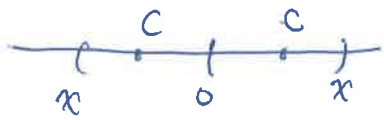
$$\approx \underline{0.1823 \cdots\cdots}$$

The first term of magnitude less than 0.01 is $\frac{(0.2)^3}{3} = 0.0026667$

§12-6

#38. $f(x)=\sin x$, $n=5$.

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}, \quad \text{for some } |c|<|x|.$$



$f^{(1)}(x)=\cos x$, $f^{(2)}(x)=-\sin x$, $f^{(3)}(x)=-\cos x$, $f^{(4)}(x)=\sin x$,

$f^{(5)}(x)=\cos x$, $f^{(6)}(x)=-\sin x$

$\therefore R_5(x)=\dfrac{-\sin c}{6!}\cdot x^6$, where c is between 0 and x.

#46. $f(x)=\sin x$

Find (a) $P_n(1)$ approximates $\sin 1$ within 0.001

(b) $P_n(2)$ approximates $\sin 2$ within 0.001

(c) $P_n(3)$ approximates $\sin 3$ within 0.001

use (12.6.6)

$$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots, \text{ for all } x\in\mathbb{R}.$$

(a) since $k=5$. $\dfrac{1}{7!}\cong 0.0002$, use $P_5(x)$

(b) since $k=9$, $\dfrac{2^{11}}{11!}\cong 0.00005$ is the first term less than 0.001, use $P_9(x)$

(c) since $k=11$, $\dfrac{3^{13}}{13!}\cong 0.0002$ is the first term less than 0.001, use $P_{11}(x)$

§12-7

#2.

$f(x) = \cos x$, $a = \frac{\pi}{3}$, $n = 4$

$f(\frac{\pi}{3}) = \frac{1}{2}$, $f^{(1)}(x) = -\sin x$, $f^{(2)}(x) = -\cos x$, $f^{(3)}(x) = \sin x$, $f^{(4)}(x) = \cos x$,

$f^{(5)}(x) = -\sin x$, $f^{(1)}(\frac{\pi}{3}) = \frac{-\sqrt{3}}{2}$, $f^{(2)}(\frac{\pi}{3}) = \frac{-1}{2}$, $f^{(3)}(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $f^{(4)}(\frac{\pi}{3}) = \frac{1}{2}$

$$P_4(x) = f(\tfrac{\pi}{3}) + \frac{f^{(1)}(\frac{\pi}{3})}{1!}(x-\tfrac{\pi}{3}) + \frac{f^{(2)}(\frac{\pi}{3})}{2!}(x-\tfrac{\pi}{3})^2 + \frac{f^{(3)}(\frac{\pi}{3})}{3!}(x-\tfrac{\pi}{3})^3 + \frac{f^{(4)}(\frac{\pi}{3})}{4!}(x-\tfrac{\pi}{3})^4$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}(x-\tfrac{\pi}{3}) - \frac{1}{4}(x-\tfrac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x-\tfrac{\pi}{3})^3 + \frac{1}{48}(x-\tfrac{\pi}{3})^4$$

$$R_4(x) = \frac{-f^{(5)}(c)}{(4+1)!}\cdot(x-\tfrac{\pi}{3})^5 = \frac{-\sin c}{5!}(x-\tfrac{\pi}{3})^5$$, where c is between $\frac{\pi}{3}$ and x.

8.

$g(x) = x^4 - x^3 + x^2 - x + 1$ in power of $x-2$. , $g(2) = 11$

$g^{(1)}(x) = 4x^3 - 3x^2 + 2x - 1$ $\quad g^{(1)}(2) = 23$

$g^{(2)}(x) = 12x^2 - 6x + 2$ $\quad g^{(2)}(2) = 38$

$g^{(3)}(x) = 24x - 6$ $\quad g^{(3)}(2) = 42$

$g^{(4)}(x) = 24$ $\quad g^{(4)}(2) = 24$

$x \in (-\infty, \infty)$

$$\Rightarrow g(x) = g(2) + \frac{g^{(1)}(2)}{1!}(x-2) + \frac{g^{(2)}(2)}{2!}(x-2)^2 + \frac{g^{(3)}(2)}{3!}(x-2)^3 + \frac{g^{(4)}(2)}{4!}(x-2)^4 = 11 + 23(x-2) + 19(x-2)^2 + 7(x-2)^3 + (x-2)^4$$

§12-7

\# 6. $f(x)=\cos(\pi x)$, $a=\frac{1}{2}$, $n=4$. $f(\frac{1}{2})=0$

$f^{(1)}(x)=-\pi\sin(\pi x)$, $f^{(2)}(x)=-\pi^2\cos(\pi x)$, $f^{(3)}(x)=\pi^3\sin(\pi x)$, $f^{(4)}(x)=\pi^4\cos(\pi x)$,

$f^{(5)}(x)=-\pi^5\sin(\pi x)$, $f^{(1)}(\frac{1}{2})=-\pi$, $f^{(2)}(\frac{1}{2})=0$, $f^{(3)}(\frac{1}{2})=\pi^3$, $f^{(4)}(\frac{1}{2})=0$

$$P_4(x)=f(\tfrac{1}{2})+\frac{f^{(1)}(\frac{1}{2})}{1!}(x-\tfrac{1}{2})+\frac{f^{(2)}(\frac{1}{2})}{2!}(x-\tfrac{1}{2})^2+\frac{f^{(3)}(\frac{1}{2})}{3!}(x-\tfrac{1}{2})^3+\frac{f^{(4)}(\frac{1}{2})}{4!}(x-\tfrac{1}{2})^4$$

$$=0+\frac{-\pi}{1}(x-\tfrac{1}{2})+\frac{0}{2!}(x-\tfrac{1}{2})^2+\frac{\pi^3}{6}(x-\tfrac{1}{2})^3+\frac{0}{24}\cdot(x-\tfrac{1}{2})^4$$

$$=-\pi(x-\tfrac{1}{2})+\frac{\pi^3}{6}(x-\tfrac{1}{2})^3$$

$$R_4(x)=\frac{f^{(5)}(c)}{(4+1)!}(x-\tfrac{1}{2})^5=\frac{-\pi^5\cdot\sin(\pi c)}{5!}(x-\tfrac{1}{2})^5,$$

where c is between $\frac{1}{2}$ and x.

12,

$g(x) = (b+x)^{-1}$ in power of $x-a$, $a \neq -b$, $(b+a \neq 0)$

$$= \frac{1}{b+x}$$

$$= \frac{1}{b+a+x-a} = \frac{1}{b+a} \cdot \frac{1}{1+\boxed{\frac{x-a}{b+a}}}$$

$$= \frac{1}{b+a} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \boxed{\left(\frac{x-a}{b+a}\right)}^k$$, since $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k \cdot x^k$, $-1 < x < 1$. $|x| < 1$

where, $\left|\frac{x-a}{a+b}\right| < 1$.

$$= \frac{1}{b+a} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(b+a)^k} \cdot (x-a)^k$$, $|x-a| < |a+b|$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(b+a)^{k+1}} \cdot (x-a)^k$$, $a - |a+b| < x < a + |a+b|$.

20. $g(x) = \sin(\pi x)$ in powers of $x-1$.

$$g(x) = \sin(\pi x) = -\sin(\pi x - \pi)$$

use (12.6.6)

$$= -\sin(\pi(x-1))$$

since $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$, for all $x \in \mathbb{R}$.

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot [\pi(x-1)]^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot \pi^{2k+1}}{(2k+1)!} \cdot (x-1)^{2k+1}, \quad \text{for all } x \in \mathbb{R}.$$

26.

$g(x) = \ln(x^2)$ in power of $x-1$.

$$g(x) = \ln(x^2)$$

$$= 2\ln x$$

use (12.6.8)

since $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot x^k$, $-1 < x \le 1$.

$$= 2\ln(1 + x-1)$$

$$= 2 \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot (x-1)^k, \quad -1 < x-1 \le 1$$

$$0 < x \le 2$$

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$g(x) = (1+2x)^{-4}$ in powers of $x-2$.

$g^{(1)}(x) = -4\,(1+2x)^{-5} \cdot 2$

$g^{(2)}(x) = -4 \cdot (-5) \cdot (1+2x)^{-6} \cdot 2^2$

$g^{(3)}(x) = (-1)^3 \cdot 4 \cdot 5 \cdot 6 \cdot (1+2x)^{-7} \cdot 2^3$

$g^{(4)}(x) = (-1)^4 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (1+2x)^{-8} \cdot 2^4$

$\vdots$

$$g^{(n)}(x) = (-1)^n \cdot \frac{(n+3)!}{3!} \cdot (1+2x)^{-n-4} \cdot 2^n$$

$$g^{(n)}(2) = (-1)^n \cdot \frac{(n+3)!}{3!} \cdot 5^{-n-4} \cdot 2^n = (-1)^n \cdot \frac{(n+3)!}{3!} \cdot \frac{2^n}{5^{n+4}}$$

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(2)}{k!} \cdot (x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \frac{(k+3)!}{3!} \cdot \frac{2^k}{5^{k+4}}}{k!} \cdot (x-2)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{3} \cdot (k+1)(k+2)(k+3) \cdot \frac{2^{k-1}}{5^{k+4}} \cdot (x-2)^k$$