HW 1 : §14-3 : 4, 8, 9, 16, 24, 36, 40

§14-4 : 4, 9, 12, 18, 20, 21, 23, 24

V

§14-3

※ 4. $r(t) = (t+1)\vec{i} + (t^2+1)\vec{j} + (t^3+1)\vec{k}$, at $P(1,1,1)$

$r'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$

as $t=0$, $r(0) = \vec{i}+\vec{j}+\vec{k} = (1,1,1) = P$ and $r'(0) = \vec{i}$

tangent line : $R(u) = r(0) + r'(0)\cdot u$

$= (\vec{i}+\vec{j}+\vec{k}) + u\cdot\vec{i}$

$= (1+u)\vec{i} + \vec{j} + \vec{k}$

※ 8. $r(t) = t\sin t\,\vec{i} + t\cos t\,\vec{j} + 2t\vec{k}$, $t = \frac{\pi}{2}$

$r'(t) = (\sin t + t\cos t)\vec{i} + (\cos t - t\sin t)\vec{j} + 2\vec{k}$

$r(\frac{\pi}{2}) = \frac{\pi}{2}\vec{i} + \pi\vec{k}$ and $r'(\frac{\pi}{2}) = \vec{i} - \frac{\pi}{2}\vec{j} + 2\vec{k}$

tangent line : $R(u) = r(\frac{\pi}{2}) + r'(\frac{\pi}{2})\cdot u$

$= (\frac{\pi}{2}\vec{i} + \pi\vec{k}) + u(\vec{i} - \frac{\pi}{2}\vec{j} + 2\vec{k})$

$= (\frac{\pi}{2}+u)\vec{i} - \frac{\pi}{2}u\vec{j} + (\pi+2u)\vec{k}$

9. Show that $r(t) = at\vec{i} + bt^2\vec{j}$ parametrizes a parabola.

Find an equation in x and y for this parabola.

<pf>

Let $x(t) = at$

$y(t) = bt^2$

$$\Rightarrow x^2(t) = a^2t^2 = \frac{a^2}{b}\cdot bt^2 = \frac{a^2}{b}\cdot y(t)$$

$$\Rightarrow bx^2 = a^2y \text{ (parabola)}$$

$\blacksquare$

16. $r_1(t) = e^{-t}\vec{i} + \cos t\,\vec{j} + (t^2+4)\vec{k}$, $r_2(u) = (2+u)\vec{i} + u^4\vec{j} + 4u^2\vec{k}$

$P(1,1,4)$

as $t=0$, $r_1(0) = \vec{i} + \vec{j} + 4\vec{k} = P(1,1,4)$

$u=-1$, $r_2(-1) = \vec{i} + \vec{j} + 4\vec{k} = P(1,1,4)$

$r_1'(t) = -e^{-t}\vec{i} - \sin t\,\vec{j} + 2t\vec{k}$, $r_2'(u) = \vec{i} + 4u^3\vec{j} + 8u\vec{k}$

$r_1'(0) = -\vec{i}$ $\qquad$ $r_2'(-1) = \vec{i} - 4\vec{j} - 8\vec{k}$

$$r_1'(0)\cdot r_2'(-1) = \|r_1'(0)\|\cdot\|r_2'(-1)\|\cdot\cos\theta$$

$$\cos\theta = \frac{1\times(-1) + 0\times(-4) + 0\times(-8)}{\sqrt{(-1)^2+0^2+0^2}\cdot\sqrt{1^2+(-4)^2+(-8)^2}} = \frac{-1}{9}$$

$$\theta = \cos^{-1}\left(\frac{-1}{9}\right) \approx 1.68 \text{ (rad)} \approx 96.4^\circ$$

§ 14-3

12

*24. $r(t) = \sin t\,\vec{i} - 2\cos t\,\vec{j}$, $t = \frac{\pi}{3}$

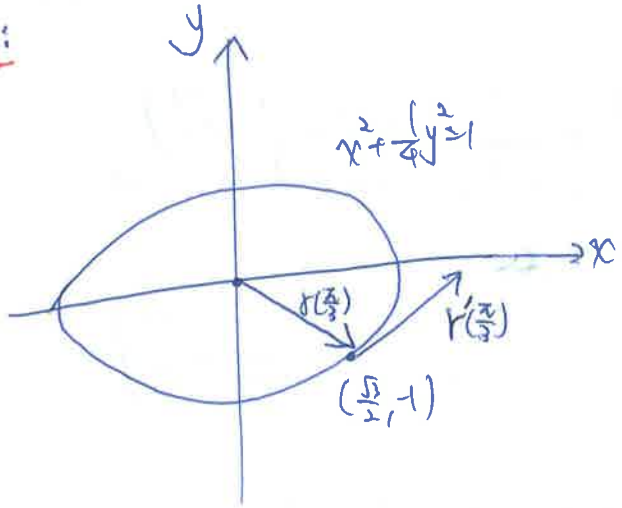
$$r\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\vec{i} - \vec{j} = \left(\frac{\sqrt{3}}{2}, -1\right)$$

Let $x = \sin t$, $y = -2\cos t$

$\Rightarrow x^2 = \sin^2 t$, $y^2 = 4\cos^2 t$, $\frac{1}{4}y^2 = \cos^2 t \Rightarrow x^2 + \frac{1}{4}y^2 = 1$ (ellipse)

$r'(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$, $r'\left(\frac{\pi}{3}\right) = \frac{1}{2}\vec{i} + \sqrt{3}\vec{j}$

graph:



*36 $r(t) = t\vec{i} + t^2\vec{j} + 2t^2\vec{k}$, $t = 1$

$r'(t) = \vec{i} + 2t\vec{j} + 4t\vec{k}$, $\|r'(t)\| = \sqrt{1+4t^2+16t^2} = \sqrt{1+20t^2}$

(unit tangent vector) $T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{\vec{i} + 2t\vec{j} + 4t\vec{k}}{\sqrt{1+20t^2}}$ $\quad T(1) = \frac{1}{\sqrt{21}}(\vec{i} + 2\vec{j} + 4\vec{k})$

$$T'(t) = \left(\frac{1}{\sqrt{1+20t^2}}\right)'\vec{i} + \left(\frac{2t}{\sqrt{1+20t^2}}\right)'\vec{j} + \left(\frac{4t}{\sqrt{1+20t^2}}\right)'\vec{k}$$

$$= \frac{\frac{1}{2}\cdot(1+20t^2)^{\frac{-1}{2}}\cdot 40t}{1+20t^2}\vec{i} + \frac{2\sqrt{1+20t^2} - 2t\cdot\frac{1}{2}\cdot(1+20t^2)^{\frac{-1}{2}}\cdot 40t}{1+20t^2}\vec{j}$$

$$+ \frac{4\sqrt{1+20t^2} - 4t\cdot\frac{1}{2}(1+20t^2)^{\frac{-1}{2}}\cdot 40t}{1+20t^2}\vec{k}$$

$$T'(t) = \frac{-20t}{(1+20t^2)^{3/2}}\vec{i} + \frac{2+40t^2-40t^2}{(1+20t^2)^{3/2}}\vec{j} + \frac{4+80t^2-80t^2}{(1+20t^2)^{3/2}}\vec{k}$$

$$= \frac{-20t}{(1+20t^2)^{3/2}}\vec{i} + \frac{2}{(1+20t^2)^{3/2}}\vec{j} + \frac{4}{(1+20t^2)^{3/2}}\vec{k}$$

$$\|T'(t)\| = \sqrt{\frac{400t^2+20}{(1+20t^2)^3}} = \frac{\sqrt{20}}{1+20t^2}$$

(principal normal)

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{-20t}{\sqrt{20+400t^2}}\vec{i} + \frac{2}{\sqrt{20+400t^2}}\vec{j} + \frac{4}{\sqrt{20+400t^2}}\vec{k}$$

$$N(1) = \frac{-20}{\sqrt{420}}\vec{i} + \frac{2}{\sqrt{420}}\vec{j} + \frac{4}{\sqrt{420}}\vec{k} = \frac{1}{\sqrt{105}}(-10\vec{i}+\vec{j}+2\vec{k})$$

$$T(1)\times N(1) = \left(\frac{1}{\sqrt{21}}\vec{i}+\frac{2}{\sqrt{21}}\vec{j}+\frac{4}{\sqrt{21}}\vec{k}\right)\times\left(\frac{-10}{\sqrt{105}}\vec{i}+\frac{1}{\sqrt{105}}\vec{j}+\frac{2}{\sqrt{105}}\vec{k}\right)$$

$$= \begin{vmatrix} \frac{2}{\sqrt{21}} & \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{105}} & \frac{2}{\sqrt{105}} \end{vmatrix}\vec{i} + \begin{vmatrix} \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} \\ \frac{2}{\sqrt{105}} & \frac{-10}{\sqrt{105}} \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{1}{\sqrt{21}} & \frac{2}{\sqrt{21}} \\ \frac{-10}{\sqrt{105}} & \frac{1}{\sqrt{105}} \end{vmatrix}\vec{k}$$

$$= \frac{-42}{21\sqrt{5}}\vec{j} + \frac{21}{21\sqrt{5}}\vec{k}$$

$$= \frac{1}{\sqrt{5}}(-2\vec{j}+\vec{k})$$

Osculating plane at $r(1) = (1,1,2)$:

$$0\cdot(x-1) + \frac{-2}{\sqrt{5}}(y-1) + \frac{1}{\sqrt{5}}(z-2) = 0$$

$$-2y+z=0$$

§14-3

13.

※40. $\gamma(t)=\cos 3t\,\vec{i}+t\,\vec{j}-\sin 3t\,\vec{k}$, $t=\frac{\pi}{3}$

$r'(t)=-3\sin 3t\,\vec{i}+\vec{j}-3\cos 3t\,\vec{k}$

$\|r'(t)\|=\sqrt{10}$.

$T(t)=\frac{-3}{\sqrt{10}}\sin 3t\,\vec{i}+\frac{1}{\sqrt{10}}\vec{j}-\frac{3}{\sqrt{10}}\cos 3t\,\vec{k}$ $T(\frac{\pi}{3})=\frac{1}{\sqrt{10}}(\vec{j}+3\vec{k})$

$T'(t)=\frac{-9}{\sqrt{10}}\cos 3t\,\vec{i}+\frac{9}{\sqrt{10}}\sin 3t\,\vec{k}$

$\|T'(t)\|=\frac{9}{\sqrt{10}}$

$N(t)=-\cos 3t\,\vec{i}+\sin 3t\,\vec{k}$ $N(\frac{\pi}{3})=\vec{i}$

$T(\frac{\pi}{3})\times N(\frac{\pi}{3})=\frac{1}{\sqrt{10}}(\vec{j}+3\vec{k})\times\vec{i}=\frac{-1}{\sqrt{10}}\vec{k}+\frac{3}{\sqrt{10}}\vec{j}=\frac{1}{\sqrt{10}}(3\vec{j}-\vec{k})$

osculating plane at $\gamma(\frac{\pi}{3})=(-1,\frac{\pi}{3},0)$:

$$0\cdot(x-(-1))+\frac{3}{\sqrt{10}}(y-\frac{\pi}{3})+\frac{-1}{\sqrt{10}}(z-0)=0$$

$$3y-\pi-z=0$$

$$3y-z-\pi=0$$

§ 14-4

※4.

$$r(t) = t\vec{i} + \frac{2}{3}\sqrt{2}\, t^{\frac{3}{2}}\vec{j} + \frac{1}{2}t^2\vec{k} \quad , 0 \leq t \leq 2$$

$$r'(t) = \vec{i} + \sqrt{2}\cdot t^{\frac{1}{2}}\vec{j} + t\vec{k} \qquad \|r'(t)\| = \sqrt{1+2t+t^2} = |t+1| = t+1$$

$$L = \int_0^2 t+1\, dt = \frac{1}{2}t^2 + t \Big|_0^2 = 4$$

※9.

$$r(t) = e^t\cos t\,\vec{i} + e^t\cdot\sin t\,\vec{j} \quad , 0 \leq t \leq \pi$$

$$r'(t) = (-e^t\sin t + e^t\cos t)\vec{i} + (e^t\sin t + e^t\cdot\cos t)\vec{j}$$

$$\|r'(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} = e^t\cdot\sqrt{2}$$

$$L = \int_0^{\pi} \sqrt{2}e^t\, dt = \sqrt{2}\cdot e^t \Big|_0^{\pi} = \sqrt{2}e^{\pi} - \sqrt{2}$$

※12. $r(t) = (t\sin t + \cos t)\vec{i} + (t\cos t - \sin t)\vec{j} + 2\vec{k} \quad , \quad 0 \leq t \leq 2,$

$$r'(t) = (\sin t + t\cos t - \sin t)\vec{i} + (\cos t - t\sin t - \cos t)\vec{j}$$

$$r'(t) = t\cos t\,\vec{i} - t\sin t\,\vec{j} \quad , \quad \|r'(t)\| = \sqrt{t^2(\cos^2 t + \sin^2 t)} = |t| = t$$

$$L = \int_0^2 t\, dt = \frac{1}{2}t^2 \Big|_0^2 = 2$$

§14-4 (4)

※18.

<pf> Let $\gamma(x) = x\vec{i} + y(x)\vec{j}$, $a \le x \le b$

$\gamma'(x) = \vec{i} + y'(x)\vec{j}$, $\|\gamma'(x)\| = \sqrt{1+(y'(x))^2}$

$$L = \int_a^b \sqrt{1+(y'(x))^2}\, dx$$

■

※20. $C_1 : \gamma(t) = (t - \ln t)\vec{i} + (t + \ln t)\vec{j}$, $1 \le t \le e$.

C_2: the graph of $y = e^x$, $0 \le x \le 1$

C_1: $\gamma'(t) = (1 - \frac{1}{t})\vec{i} + (1 + \frac{1}{t})\vec{j}$ $\|\gamma'(t)\| = \sqrt{(1-\frac{1}{t})^2 + (1+\frac{1}{t})^2} = \sqrt{2 + \frac{2}{t^2}}$

$$L(C_1) = \int_1^e \sqrt{2 + \frac{2}{t^2}}\, dt = \sqrt{2} \cdot \int_1^e \sqrt{1 + \frac{1}{t^2}}\, dt$$

C_2: $y = e^x$; $y'(x) = e^x$, use problem 18.

$$L(C_2) = \int_0^1 \sqrt{1+(e^x)^2}\, dx = \int_0^1 \sqrt{1+e^{2x}}\, dx$$

Let $t = e^x$, $x=0 \to t=1$
$dt = e^x dx$, $x=1 \to t=e$

$$L(C_2) = \int_0^1 \sqrt{1+e^{2x}}\, dx = \int_1^e \sqrt{1+t^2} \cdot \frac{1}{t}\, dt = \int_1^e \sqrt{1 + \frac{1}{t^2}}\, dt = \sqrt{2} \cdot L(C_1)$$

we get $\underline{L(C_2) = \sqrt{2} \cdot L(C_1)}$ ※

※21. $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$, $0 \le t \le 2\pi$,

<pf> $r'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\because \|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow s = \int_0^t 1\,d\ = t$ ※

$\therefore$ the parametrization is by arc length. ■

※23. $r(t) = (\sin t - t\cos t)\,\vec{i} + (\cos t + t\sin t)\,\vec{j} + \frac{1}{2}t^2\vec{k}$, $t \ge 0$,

<pf>

$r'(t) = (\cos t - \cos t + t\sin t)\,\vec{i} + (-\sin t + \sin t + t\cos t)\,\vec{j} + t\,\vec{k}$

$r'(t) = t\sin t\,\vec{i} + t\cos t\,\vec{j} + t\,\vec{k}$

$\|r'(t)\| = \sqrt{(t\sin t)^2 + (t\cos t)^2 + t^2} = \sqrt{2t^2} = \sqrt{2}\,|t| = \sqrt{2}\,t$

$\Rightarrow s = \int_0^t \sqrt{2}\,u\,du = \frac{\sqrt{2}}{2}u^2\Big|_0^t = \frac{\sqrt{2}}{2}t^2 \Rightarrow t = \left(\frac{2}{\sqrt{2}}s\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}\sqrt{s}$

$\therefore R(s) = \left(\sin(2^{\frac{1}{4}}\sqrt{s}) - 2^{\frac{1}{4}}\cdot\sqrt{s}\cdot\cos(2^{\frac{1}{4}}\cdot\sqrt{s})\right)\vec{i} + \left(\cos(2^{\frac{1}{4}}\sqrt{s}) + 2^{\frac{1}{4}}\sqrt{s}\cdot\sin(2^{\frac{1}{4}}\cdot\sqrt{s})\right)\vec{j} + \frac{s}{\sqrt{2}}\vec{k}$ ※

$s \ge 0$

§ 14-4

※24.

$$\gamma(t) = e^t \cos t \,\vec{i} + e^t \sin t \,\vec{j} \quad , \quad 0 \le t \le \pi ,$$

$$\gamma'(t) = (e^t \cos t - e^t \sin t)\,\vec{i} + (e^t \sin t + e^t \cos t)\,\vec{j}$$

$$\|\gamma'(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} = e^t \cdot \sqrt{2}$$

$$\Rightarrow S = \int_0^t \sqrt{2} e^u \, du = \sqrt{2} e^u \Big|_0^t = \sqrt{2}(e^t - 1) \Rightarrow e^t = \frac{1}{\sqrt{2}} s + 1$$

$$\Rightarrow t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\therefore R(s) = \left(\frac{s}{\sqrt{2}} + 1\right) \cdot \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \vec{i} + \left(\frac{s}{\sqrt{2}} + 1\right) \cdot \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right) \vec{j} \quad , \quad 0 \le s \le \sqrt{2}(e^{\pi} - 1)$$

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