Names and Student IDs: ______________________________

Homework 2 Calculus 2

You may assume the following integral in this homework:

$$\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx = \frac{2}{2n+1}. \tag{1}$$

1. Salas § 8.3, Problem 53

2. Let

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

   Show that

   (a)

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \ \forall m \neq n.$$

   (b)

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

   (Hint: It is helpful to observe that $\frac{d^j}{dx^j}|_{x=\pm 1}(x^2-1)^n = 0 \quad \forall j < n$. Do IBP and use (1) above.)

3. Use Problem 1 to show that

   (a)

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2n\cdot 2n}{(2n-1)(2n+1)}\frac{\int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx}{\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx}.$$

   (b)

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2n\cdot 2n}{(2n-1)(2n+1)}.$$

   That is, the integrals in part (a) converges to 1 as $n \to \infty$.

   (Hint: Note that, for $x \in [0, \frac{\pi}{2}]$, $0 \le \sin^{2n+1}x \le \sin^{2n}x \le \sin^{2n-1}x$. (why?). Integrate the inequality and use problem 1.)

4. Continue from problem 2,

   (a) show that

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \ \mathit{if}\ m < n.$$

(b) evaluate

$$\int_{-1}^{1} x^n P_n(x) dx.$$

5. Prove that

$$\int_0^x \left[\int_0^u f(t)\, dt \right] du = \int_0^x f(u)(x-u)\, du.$$

6. Probem 3, 4 of Project 8.2, pp410-411 on Salas.

7. Salas § 8.2: 5, 14, 19, 31, 36, 44, 54, 67, 77.

8. Salas § 8.3: 4, 6, 12, 16, 28, 47.

9. Salas § 8.4: 6, 11, 20, 26, 28, 35, 43.

10. (Extra Credit - 5 points) Prove (1).