

HW 2

1. Salas §8-3 #53.

(a) Use integration by parts to show that for $n > 2$,

$$\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

(b) Then show that

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$$

(c) Verify the Wallis sine formula:

for even $n \geq 2$,
$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1) \cdots \cdot 5 \cdot 3 \cdot 1}{n \cdots \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2};$$

for odd $n \geq 3$,
$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1) \cdots \cdot 4 \cdot 2}{n \cdots \cdot 5 \cdot 3}$$

(a)

(pf)

$$\int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx = -\cos x \cdot \sin^{n-1} x + \int \cos x \cdot (n-1) \cdot \sin^{n-2} x \cdot \cos x \, dx$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \cdot \sin^{n-2} x \, dx$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\Rightarrow n \int \sin^n x \, dx = -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

$$\Rightarrow \int \sin^n x \, dx = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

(b)

(pf) Use (a), then

$$\int_0^{\pi/2} \sin^n x \, dx = \left. \frac{-\cos x \cdot \sin^{n-1} x}{n} \right|_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

(c)

(pf) Use (b), then

① for n is even,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x \, dx &= \frac{n-1}{n} \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \int_0^{\frac{\pi}{2}} \sin^{n-4} x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} 1 \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.\end{aligned}$$

② for n is odd,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x \, dx &= \frac{n-1}{n} \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \int_0^{\frac{\pi}{2}} \sin^{n-4} x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \sin x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot [-\cos x]_0^{\frac{\pi}{2}} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}\end{aligned}$$



2. Let $P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$.

Show that

(a) $\int_{-1}^1 P_n(x) \cdot P_m(x) dx = 0 \quad \forall m \neq n.$ (b) $\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}.$

Hint: $\left. \frac{d^j}{dx^j} (x^2-1)^n \right|_{x=\pm 1} = 0 \quad \forall j < n.$ $\left(\left. \frac{d^j}{dx^j} (x^2-1)^n \right|_{x=\pm 1} = 0 \quad \forall j < n. \right)$

(a) \int_{-1}^1 for $m \neq n, m, n \in \mathbb{N}$, then

$$\int_{-1}^1 \left(\frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n \right) \cdot \left(\frac{1}{2^m \cdot m!} \cdot \frac{d^m}{dx^m} (x^2-1)^m \right) dx$$

$$= \frac{1}{2^n \cdot n!} \cdot \frac{1}{2^m \cdot m!} \cdot \int_{-1}^1 \frac{d^n}{dx^n} (x^2-1)^n \cdot \frac{d^m}{dx^m} (x^2-1)^m dx$$

Let $n < m$.

Then $\int_{-1}^1 \frac{d^n}{dx^n} (x^2-1)^n \cdot \frac{d^m}{dx^m} (x^2-1)^m dx$

$$= \frac{d^n}{dx^n} (x^2-1)^n \cdot \frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \Big|_{-1}^1 - \int_{-1}^1 \left(\frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \right) \cdot \left(\frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right) dx$$

$= 0$ (v Hint)

$$= (-1) \cdot \int_{-1}^1 \left(\frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \right) \cdot \left(\frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right) dx$$

$$= (-1)^2 \cdot \int_{-1}^1 \left(\frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n \right) \cdot \left(\frac{d^{m-2}}{dx^{m-2}} (x^2-1)^m \right) dx$$

$$= (-1)^m \cdot \int_{-1}^1 \left(\frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \right) \cdot (x^2-1)^m dx$$

$$\because n < m \Rightarrow 2n < n+m$$

$$\therefore \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n = 0, \text{ since } (x^2-1)^n \text{ is a polynomial of degree } 2n.$$

$$\Rightarrow \int_{-1}^1 \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \cdot (x^2-1)^m dx = 0 \text{ for } n < m.$$

$$\Rightarrow \int_{-1}^1 P_n(x) P_m(x) dx = 0 \text{ for } n < m.$$

$$\Rightarrow \int_{-1}^1 P_n(x) P_m(x) dx = 0 \text{ for } n \neq m. \quad \blacksquare$$

(b)
 <pf> $\int_{-1}^1 P_n^2(x) dx = \int_{-1}^1 \left(\frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n \right) \cdot \left(\frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n \right) dx$

$$= \frac{1}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 \frac{d^n}{dx^n} (x^2-1)^n \cdot \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$= \frac{(-1)^n}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 (x^2-1)^n \cdot \frac{d^{2n}}{dx^{2n}} (x^2-1)^n dx$$

$$\text{Now, } \frac{d^{2n}}{dx^{2n}} (x^2-1)^n = \frac{d^{2n}}{dx^{2n}} \left(\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k} \right) = \frac{d^{2n}}{dx^{2n}} (x^{2n}) = (2n)!$$

$$\Rightarrow \int_{-1}^1 P_n^2(x) dx = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} \int_{-1}^1 (x^2-1)^n \cdot (2n)! dx$$

$$= \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 (x^2-1)^n dx$$

$$= \frac{2}{2n+1} \quad (\text{see *10}) \quad \blacksquare$$

3. Use Problem 2 to show that

$$(a) \quad \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{(2n-1)} \cdot \frac{2n}{(2n+1)} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx}$$

$$(b) \quad \frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

That is, the integrals in part (a) converges to 1 as $n \rightarrow \infty$.

Hint: for $x \in [0, \frac{\pi}{2}]$, $0 \leq \sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x$ (why?)

(a)
<pf>

See *1(c), we have for each $n \in \mathbb{N}$,

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{(2n-1) \cdot (2n-3) \cdots 3 \cdot 1}{2n \cdot (2n-2) \cdots 4 \cdot 2} \cdot \frac{\pi}{2} \quad \text{and}$$

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2n \cdot (2n-2) \cdots 4 \cdot 2}{(2n+1) \cdot (2n-1) \cdots 5 \cdot 3}$$

$$\Rightarrow \frac{\pi}{2} = \frac{\frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n}{2n-1} \cdot \int_0^{\pi/2} \sin^{2n} x \, dx}{\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \cdot \frac{2n+1}{2n} \cdot \int_0^{\pi/2} \sin^{2n+1} x \, dx} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx}$$

$$\Rightarrow \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx}$$

(b)
<pf> Claim: $\lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x \, dx}{\int_0^{\pi/2} \sin^{2n+1} x \, dx} = 1$.

$$\because 0 \leq x \leq \frac{\pi}{2} \Rightarrow 0 \leq \sin x \leq 1$$

$$\therefore \sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x \text{ for each } n \in \mathbb{N}.$$

$$\Rightarrow 0 < \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n-1} x \, dx, \quad \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{2n \cdots 2}{(2n+1) \cdots 3} > 0,$$

$$\Rightarrow 1 \leq \frac{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx} \leq \frac{\int_0^{\frac{\pi}{2}} \sin^{2n-1} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}, \quad \text{for each } n \in \mathbb{N}.$$

Recall $\star 1(b)$, then we have

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{2n}{2n+1} \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx.$$

$$\Rightarrow \frac{\int_0^{\frac{\pi}{2}} \sin^{2n-1} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx} = \frac{1+2n}{2n} = 1 + \frac{1}{2n}$$

$$\therefore \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right) = 1 \quad \text{and} \quad 1 \leq \frac{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx} \leq 1 + \frac{1}{2n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx} = 1 \quad (\text{converges})$$

$$\text{Thus } \lim_{n \rightarrow \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx} \right) = \lim_{n \rightarrow \infty} \left(\frac{\pi}{2} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2} \quad (\text{converges})$$

4, Continue from Problem 2,

(a) show that $\int_{-1}^1 x^m \cdot P_n(x) dx = 0$ if $m < n$.

(b) evaluate $\int_{-1}^1 x^n \cdot P_n(x) dx$.

(a)

<pf> For $m < n$, $m, n \in \mathbb{N}$.

$$\int_{-1}^1 x^m \cdot P_n(x) dx = \int_{-1}^1 x^m \cdot \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n dx = \frac{1}{2^n \cdot n!} \int_{-1}^1 x^m \cdot \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$\Rightarrow \int_{-1}^1 x^m \cdot \frac{d^n}{dx^n} (x^2-1)^n dx = \underbrace{\left(\frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \right)}_{=0} \cdot m x^{m-1} \Big|_{-1}^1 - m \cdot \int_{-1}^1 x^{m-1} \cdot \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n dx$$

$$= (-m) \cdot \int_{-1}^1 x^{m-1} \cdot \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n dx$$

$$= (-1)^2 \cdot m \cdot (m-1) \cdot \int_{-1}^1 x^{m-2} \cdot \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n dx$$

$$= (-1)^m \cdot m! \cdot \int_{-1}^1 \frac{d^{n-m+1}}{dx^{n-m+1}} (x^2-1)^n dx$$

$$= (-1)^m \cdot m! \cdot \left(\frac{d^{n-m+1}}{dx^{n-m+1}} (x^2-1)^n \Big|_{-1}^1 \right), \quad n-m+1 \geq 0$$

$$= 0$$



(b) $\int_{-1}^1 x^n \cdot P_n(x) dx = \int_{-1}^1 x^n \cdot \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n dx$

$$= \frac{1}{2^n \cdot n!} \cdot \int_{-1}^1 x^n \cdot \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$= \frac{(-1)^n \cdot n!}{2^n \cdot n!} \cdot \int_{-1}^1 (x^2-1)^n dx$$

$$= \frac{(-1)^n \cdot n!}{2^n \cdot n!} \cdot \frac{2^{2n} \cdot (n!)^2}{(-1)^n \cdot (2n)!} \cdot \frac{2}{2n+1} \quad (\text{see } \ast 10)$$

$$= 2^n \cdot \frac{(n!)^2}{(2n)!} \cdot \frac{2}{2n+1}$$



5. Prove that $\int_0^x \left[\int_0^u f(t) dt \right] du = \int_0^x f(u) (x-u) du$.

(pf) $\int_0^x f(u) (x-u) du = x \cdot \int_0^x f(u) du - \int_0^x f(u) \cdot u du \quad (1)$

$$\int_0^x f(u) u du = u \cdot \int_a^u f(t) dt \Big|_{u=0}^{u=x} - \int_0^x \left(\int_a^u f(t) dt \right) du, \quad \begin{array}{l} s=u, ds=du \\ dv=f(u)du, \\ v = \int_a^u f(t) dt, a \in \mathbb{R} \end{array}$$

$$= x \cdot \int_a^x f(t) dt - \int_0^x \left(\int_a^u f(t) dt \right) du \quad (2)$$

(2) $\hat{=} (1)$,

$$\begin{aligned} \int_0^x f(u) \cdot (x-u) du &= x \cdot \int_0^x f(u) du - x \cdot \int_a^x f(t) dt + \int_0^x \left(\int_a^u f(t) dt \right) du \\ &= \underline{x \cdot \int_0^x f(t) dt} - x \cdot \int_a^x f(t) dt + \int_0^x \left(\int_a^u f(t) dt \right) du \\ &= \underline{x \cdot \int_0^a f(t) dt} + \int_0^x \left(\int_a^u f(t) dt \right) du \\ &= \underline{\int_0^x \left(\int_0^a f(t) dt \right) du} + \int_0^x \left(\int_a^u f(t) dt \right) du \\ &= \int_0^x \left(\int_0^u f(t) dt \right) du \end{aligned}$$

6. Problem 3.4 of Project 8-2, pp 410-411 on Salas.

Problem 3. Show that for $m \neq n$,

$$\int_0^{2\pi} \sin(mx) \cdot \cos(nx) dx = 0.$$

Problem 4. (The superposition of waves)

A function of the form

$$f(x) = a_1 \sin x + a_2 \sin(2x) + \dots + a_n \sin(nx) + b_1 \cos x + b_2 \cos(2x) + \dots + b_n \cos(nx)$$

is called a trigonometric polynomial, and the coefficients a_k, b_k is called the Fourier coefficients. Determine the a_k and b_k from $k=1$ to $k=n$.

<pt of Problem 3 >

For $m \neq n, m, n \in \mathbb{N}$.

$$\textcircled{1} \int_0^{2\pi} \sin[(m+n)x] dx = \frac{-1}{m+n} \cos[(m+n)x] \Big|_{x=0}^{x=2\pi} = 0$$

$$\textcircled{2} \text{ Since } \sin[(m+n)x] = \sin(mx + nx) = \sin(mx) \cos(nx) + \cos(mx) \sin(nx),$$

$$\text{then } \int_0^{2\pi} \sin[(m+n)x] dx = \int_0^{2\pi} \sin(mx) \cos(nx) + \cos(mx) \sin(nx) dx$$

$$= \int_0^{2\pi} \sin(mx) \cos(nx) dx + \int_0^{2\pi} \cos(mx) \sin(nx) dx = 0.$$

$$\Rightarrow \int_0^{2\pi} \sin(mx) \cos(nx) dx = - \int_0^{2\pi} \cos(mx) \sin(nx) dx$$

$$= \cos(mx) \cdot \frac{-1}{n} \cos(nx) \Big|_0^{2\pi} - \int_0^{2\pi} \frac{1}{n} \cos(nx) \cdot m \cdot \sin(mx) dx$$

$$= \frac{-m}{n} \int_0^{2\pi} \cos(nx) \sin(mx) dx.$$

$$\Rightarrow \left(1 + \frac{m}{n}\right) \cdot \int_0^{2\pi} \sin(mx) \cos(nx) dx = 0$$

Since $\frac{m+n}{n} \neq 0$, then $\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0$ for $m \neq n, m, n \in \mathbb{N}$.

<pf of Problem 4>

$$f(x) = a_1 \sin x + a_2 \sin(2x) + \dots + a_n \sin(nx) + b_1 \cos x + b_2 \cos(2x) + \dots + b_n \cos(nx)$$

Given $k \in \mathbb{N}, 1 \leq k \leq n$.

$$\int_0^{2\pi} f(x) \cdot \sin(kx) dx = \int_0^{2\pi} \left(\sum_{i=1}^n a_i \sin(ix) + b_i \cos(ix) \right) \cdot \sin(kx) dx \quad (1)$$

$$\int_0^{2\pi} f(x) \cdot \cos(kx) dx = \int_0^{2\pi} \left(\sum_{i=1}^n a_i \sin(ix) + b_i \cos(ix) \right) \cdot \cos(kx) dx \quad (2)$$

Now, for $m \neq n, m, n \in \mathbb{N}$, claim: $\int_0^{2\pi} \sin(mx) \sin(nx) dx = 0$

$$\text{and } \int_0^{2\pi} \cos(mx) \cos(nx) dx = 0.$$

$$\textcircled{1} \int_0^{2\pi} \cos[(m+n)x] dx = \frac{1}{m+n} \sin[(m+n)x] \Big|_0^{2\pi} = 0.$$

\textcircled{2} Since $\cos[(m+n)x] = \cos(mx) \cos(nx) - \sin(mx) \sin(nx)$, then

$$\int_0^{2\pi} \cos[(m+n)x] dx = \int_0^{2\pi} \cos(mx) \cos(nx) dx - \int_0^{2\pi} \sin(mx) \sin(nx) dx = 0$$

$$\Rightarrow \int_0^{2\pi} \cos(mx) \cos(nx) dx = \int_0^{2\pi} \sin(mx) \sin(nx) dx.$$

$$\begin{aligned} \Rightarrow \int_0^{2\pi} \cos(mx) \cos(nx) dx &= \int_0^{2\pi} \sin(mx) \sin(nx) dx \\ &= \sin(mx) \cdot \frac{-1}{n} \cos(nx) \Big|_0^{2\pi} + \int_0^{2\pi} \frac{1}{n} \cos(nx) \cdot m \cos(mx) dx \\ &= \frac{m}{n} \int_0^{2\pi} \cos(nx) \cos(mx) dx \end{aligned}$$

$$\Rightarrow \left(\frac{m+n}{n}\right) \int_0^{2\pi} \cos(nx) \cos(mx) dx = 0$$

Since $\frac{m+n}{n} \neq 0$, then $\int_0^{2\pi} \cos(nx) \cos(mx) dx = 0$ for $m \neq n, m, n \in \mathbb{N}$. — (*)

$\Rightarrow \int_0^{2\pi} \sin(mx) \sin(nx) dx = 0$ for $m \neq n, m, n \in \mathbb{N}$. — (**)

Use Problem 3, (*), and (**), then

$$\text{for (1)} \Rightarrow \int_0^{2\pi} f(x) \sin(kx) dx = \int_0^{2\pi} a_k \sin^2(kx) dx + \int_0^{2\pi} b_k \sin(kx) \cos(kx) dx \quad (3)$$

$$\text{for (2)} \Rightarrow \int_0^{2\pi} f(x) \cos(kx) dx = \int_0^{2\pi} a_k \sin(kx) \cos(kx) dx + \int_0^{2\pi} b_k \cos^2(kx) dx \quad (4)$$

$$\text{Now, } \int_0^{2\pi} \sin(kx) \cos(kx) dx = \int_0^{2\pi} \frac{1}{2} \sin(2kx) dx = \frac{-1}{4k} \cos(2kx) \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2(kx) dx = \int_0^{2\pi} \frac{1 - \cos(2kx)}{2} dx = \frac{1}{2}x - \frac{1}{4k} \sin(2kx) \Big|_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \cos^2(kx) dx = \int_0^{2\pi} \frac{1 + \cos(2kx)}{2} dx = \frac{1}{2}x + \frac{1}{4k} \sin(2kx) \Big|_0^{2\pi} = \pi$$

$$\Rightarrow \int_0^{2\pi} f(x) \sin(kx) dx = a_k \cdot \pi \quad \text{and} \quad \int_0^{2\pi} f(x) \cos(kx) dx = b_k \cdot \pi$$

$$\Rightarrow a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \quad 1 \leq k \leq n.$$

7. Salas §8-2 * 5, 14, 19, 31, 36, 44, 54, 67, 77.

5.

$$\int x^2 \cdot e^{-x} dx = -x^2 \cdot e^{-x} + 2 \int e^{-x} \cdot x dx = -x^2 \cdot e^{-x} + 2 [-x e^{-x} + \int e^{-x} dx]$$

$$u = x^2, dv = e^{-x} dx$$

$$u = x, dv = e^{-x} dx$$

$$du = 2x dx, v = -e^{-x}$$

$$du = 2x dx, v = -e^{-x}$$

$$\Rightarrow \int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$\Rightarrow \int_0^1 x^2 e^{-x} dx = -e^{-1} - 2e^{-1} - 2e^{-1} + 2 = 2 - 5e^{-1} \quad *$$

14.

$$\int x(x+5)^{-14} dx = \int (u-5) \cdot u^{-14} du = \int u^{-13} - 5u^{-14} du$$

$$u = x+5$$

$$du = dx$$

$$= \frac{1}{-12} u^{-12} + \frac{5}{-13} u^{-13} + C$$

$$= \frac{1}{12} (x+5)^{-12} + \frac{5}{13} (x+5)^{-13} + C \quad *$$

19.

$$\int x \cos(\pi x) dx = \frac{1}{\pi} x \cdot \sin(\pi x) - \frac{1}{\pi} \int \sin(\pi x) dx$$

$$u = x, dv = \cos(\pi x) dx$$

$$du = dx, v = \frac{1}{\pi} \sin(\pi x)$$

$$= \frac{x}{\pi} \sin(\pi x) + \frac{1}{\pi^2} \cos(\pi x) + C$$

$$\Rightarrow \int_0^{\frac{1}{2}} x \cos(\pi x) dx = \frac{1}{2\pi} \sin\left(\frac{\pi}{2}\right) + \frac{1}{\pi^2} \cos\left(\frac{\pi}{2}\right) - \frac{1}{\pi^2} = \frac{1}{2\pi} - \frac{1}{\pi^2} \quad *$$

$$31. \int \sin^{-1}(2x) dx = \frac{1}{2} \int \sin^{-1}(u) du = \frac{1}{2} (u \cdot \sin^{-1}(u) + \sqrt{1-u^2}) + C$$

$$u=2x$$

$$du=2 dx$$

$$= \frac{1}{2} \cdot (2x \cdot \sin^{-1}(2x) + \sqrt{1-4x^2}) + C$$

$$\Rightarrow \int_0^{\frac{1}{4}} \sin^{-1}(2x) dx = \frac{1}{2} \left[\frac{1}{2} \sin^{-1}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} \right] - \frac{1}{2} = \frac{1}{4} \times \frac{\pi}{6} + \frac{\sqrt{3}}{4} - \frac{2}{4}$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}-2}{4} \quad \times$$

36.

$$\int x \sinh 2x^2 dx = \int \frac{1}{4} \sinh(u) du = \frac{1}{4} \cosh(u) + C = \frac{1}{4} \cosh(2x^2) + C$$

$$u=2x^2$$

$$du=4x dx$$

$$\Rightarrow \int_{-1}^1 x \cdot \sinh 2x^2 dx = \frac{1}{4} \cosh 2 - \frac{1}{4} \cosh 2 = 0 \quad \times$$

$$44. \int e^{ax} \cos bx dx = \frac{1}{b} \sin bx \cdot e^{ax} - \frac{a}{b} \int e^{ax} \sin bx dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[\frac{-1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx dx \right]$$

$$\begin{cases} u=e^{ax} & dv=\cos bx dx \\ du=ae^{ax} dx & v=\frac{1}{b} \sin bx \end{cases}$$

$$\begin{cases} u=e^{ax} & dv=\sin bx dx \\ du=ae^{ax} dx & v=-\frac{1}{b} \cos bx \end{cases}$$

$$\Rightarrow \int e^{ax} \cos bx dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx dx$$

$$\Rightarrow \int e^{ax} \cos bx dx = \frac{e^{ax} \cdot a \cos bx + e^{ax} \cdot b \sin bx}{a^2 + b^2} + C \quad \times$$

54,

$$f(x) = e^{-x}, \quad x \in [0, 1]$$

$$\text{formula: } \bar{x}A = \int_a^b x f(x) dx, \quad \bar{y}A = \int_a^b \frac{1}{2} (f(x))^2 dx, \quad A = \int_a^b f(x) dx$$

$$A = \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$$

$$\bar{x} \cdot A = \int_0^1 x e^{-x} dx = -x e^{-x} \Big|_0^1 + \int_0^1 e^{-x} dx = -e^{-1} + 1 - \frac{1}{e} = 1 - \frac{2}{e}$$

$$\begin{aligned} u &= x & dv &= e^{-x} dx \\ du &= dx & v &= -e^{-x} dx \end{aligned} \quad \Rightarrow \quad \bar{x} = \frac{e-2}{e-1}$$

$$\bar{y} \cdot A = \int_0^1 \frac{1}{2} e^{-2x} dx = \frac{-1}{4} e^{-2x} \Big|_0^1 = \frac{1}{4} e^{-2} + \frac{1}{4} \quad \Rightarrow \quad \bar{y} = \frac{e+1}{4e}$$

67,
(cpts)Let $n \in \mathbb{N}$, $a \neq 0$.

$$\Rightarrow \int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx,$$

$$u = x^n \quad dv = e^{ax} dx$$

$$du = n x^{n-1} dx \quad v = \frac{1}{a} e^{ax}$$



77

(pf)

$$\int_a^b f(x) g''(x) dx = f(x) g'(x) \Big|_a^b - \int_a^b g'(x) f'(x) dx = - \int_a^b g'(x) f'(x) dx \quad \text{--- (1)}$$

$$u = f(x) \quad dv = g''(x) dx$$

$$du = f'(x) dx \quad v = g'(x)$$

$$\int_a^b f'(x) g(x) dx = g(x) f'(x) \Big|_a^b - \int_a^b g(x) f''(x) dx = - \int_a^b g(x) f''(x) dx \quad \text{--- (2)}$$

$$u = f'(x) \quad dv = g(x) dx$$

$$du = f''(x) dx \quad v = g(x)$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \int_a^b f(x) g''(x) dx = \int_a^b g(x) f''(x) dx.$$

8. Salas §8-3 * 4, 6, 12, 16, 28, 47.

$$4. \int \cos^3 x \, dx = \int (1 - \sin^2 x) d(\sin x) = \sin x - \frac{1}{3} \sin^3 x + C$$

*

$$6. \int \sin^3 x \cos^2 x \, dx = \int (1 - u^2) u^2 (-du) = \int u^4 - u^2 \, du$$

$$u = \sin x$$

$$du = \cos x \, dx$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

*

$$12. \int \cot^3 x \, dx = \int (\csc^2 x - 1) \cot x \, dx = \int \cot x \cdot \csc^2 x \, dx - \int \cot x \, dx$$

$$u = \cot x$$

$$du = -\csc^2 x \, dx$$

$$= -\int u \, du - \int \frac{\cos x \, dx}{\sin x}$$

$$= -\frac{1}{2} u^2 - \ln |\sin x| + C$$

$$= -\frac{1}{2} \cot^2 x - \ln |\sin x| + C$$

*

$$16. \int \cos 2x \sin 3x \, dx = \int \frac{1}{2} [\sin 5x + \sin x] \, dx = \frac{1}{2} \int \sin 5x \, dx + \frac{1}{2} \int \sin x \, dx$$

$$= -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos 2x \sin 3x \, dx = -\frac{1}{10} \cos \frac{5\pi}{2} + \frac{1}{2} \cos \frac{\pi}{2} + \frac{1}{10} + \frac{1}{2}$$

$$= \frac{6}{10} = \frac{3}{5}$$

*

$$28. \int \sec^4 3x \, dx = \int (1 + \tan^2 3x) \sec^2 3x \, dx = \int \frac{1}{3} (1 + u^2) \, du$$

$$u = \tan 3x$$

$$du = 3 \sec^2 3x \, dx$$

$$= \int \frac{1}{3} + \frac{1}{3} u^2 \, du = \frac{1}{3} u + \frac{1}{9} u^3 + C$$

$$= \frac{1}{3} \tan 3x + \frac{1}{9} \tan^3 3x + C$$

~~*~~

47

$$y = \sin^4 x, \quad x \in [0, \pi]$$

$$V = \int_0^{\pi} \pi (\sin^2 x)^2 \, dx = \int_0^{\pi} \pi \cdot \sin^4 x \, dx$$

$$\int \sin^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 \, dx = \int \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \, dx$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{4} \int \cos^2 2x \, dx$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{4} \left[\int \frac{1 + \cos 4x}{2} \, dx \right]$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{32} \sin 4x + C$$

$$\Rightarrow V = \int_0^{\pi} \pi \cdot \sin^4 x \, dx = \pi \cdot \left[\frac{3}{8} \cdot \pi \right] = \frac{3}{8} \pi^2$$

~~*~~

9. Solas 5f-4 * 4, 6, 11, 20, 26, 28, 35, 43.

$$4. \int \frac{x}{\sqrt{4-x^2}} dx = \int \frac{2 \sin \theta}{2 \cos \theta} \cdot 2 \cos \theta d\theta = \int 2 \sin \theta d\theta$$

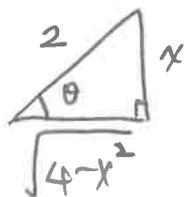
$$x = 2 \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$dx = 2 \cos \theta d\theta$$

$$= -2 \cos \theta + C$$

$$= -\sqrt{4-x^2} + C$$

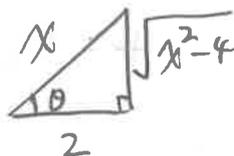
~~✗~~



$$6. \int \frac{x^2}{\sqrt{x^2-4}} dx = \int \frac{4 \sec^2 \theta}{2 \tan \theta} \cdot 2 \sec \theta \cdot \tan \theta d\theta = \int 4 \sec^3 \theta d\theta$$

$$x = 2 \sec \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$dx = 2 \sec \theta \tan \theta d\theta$$



$$\because \int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \cdot \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$

$$\Rightarrow \int \frac{x^2}{\sqrt{x^2-4}} dx = 2 \sec \theta \cdot \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{1}{2} x \sqrt{x^2-4} + 2 \ln \left| \frac{x}{2} + \frac{\sqrt{x^2-4}}{2} \right| + C$$

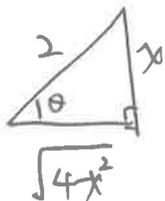
~~✗~~

$$11. \int x \sqrt{4-x^2} dx = \int 2 \sin \theta \cdot 2 \cos \theta \cdot 2 \cos \theta d\theta$$

$$x = 2 \sin \theta, 0 \leq \theta \leq \frac{\pi}{2} = 8 \int \sin \theta \cos^2 \theta d\theta$$

$$dx = 2 \cos \theta d\theta = 8 \int \cos^2 \theta d(-\cos \theta)$$

$$= -\frac{8}{3} \cos^3 \theta + C$$



$$= \frac{-8}{3} \cdot \frac{(\sqrt{4-x^2})^3}{8} + C = \frac{-1}{3} \cdot (\sqrt{4-x^2})^3 + C \quad *$$

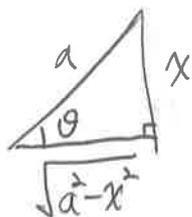
20.

$$\int \frac{1}{x \sqrt{a^2-x^2}} dx = \int \frac{1}{a^2 \sin^2 \theta \cdot a \cos \theta} \cdot a \cos \theta d\theta = \frac{1}{a^2} \int \csc^2 \theta d\theta$$

$$= \frac{-1}{a^2} \cot \theta + C$$

$$x = a \sin \theta, 0 \leq \theta \leq \frac{\pi}{2}$$

$$dx = a \cos \theta d\theta$$



$$= \frac{-1}{a^2} \cdot \frac{\sqrt{a^2-x^2}}{x} + C \quad *$$

$$26. \int \frac{e^x}{\sqrt{9-e^{2x}}} dx = \int \frac{1}{\sqrt{9-u^2}} du = \int \frac{1}{3 \cos \theta} \cdot 3 \cos \theta d\theta$$

$$= \int 1 d\theta = \theta + C$$

$$\begin{cases} u = e^x \\ du = e^x dx \end{cases}$$

$$= \sin^{-1}\left(\frac{u}{3}\right) + C$$

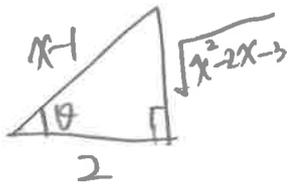
$$\begin{cases} u = 3 \sin \theta \\ du = 3 \cos \theta d\theta \end{cases}$$

$$= \sin^{-1}\left(\frac{e^x}{3}\right) + C \quad *$$

$$28. \int \frac{1}{\sqrt{x^2-2x-3}} dx = \int \frac{1}{\sqrt{(x-1)^2-4}} dx = \int \frac{1}{2+\tan\theta} \cdot 2\sec\theta \tan\theta d\theta$$

$$x-1 = 2\sec\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$dx = 2\sec\theta \tan\theta d\theta$$



$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x-1}{2} + \frac{\sqrt{x^2-2x-3}}{2}\right| + C$$

✱

$$35. \int \sec^2 x dx = x \sec^2 x - \int \frac{x}{x \cdot \sqrt{x^2-1}} dx = x \sec^2 x - \int \frac{1}{\sqrt{x^2-1}} dx$$

$$\text{Let } x = \sec u, \quad 0 \leq u \leq \frac{\pi}{2} \quad \Rightarrow \quad u = \sec^{-1} x, \quad du = dx$$

$$\Rightarrow x > 1 \text{ and } u = \sec^{-1} x. \quad du = \frac{x}{x \sqrt{x^2-1}} dx, \quad v = x$$

$$\text{Now, } \int \frac{x}{|x| \cdot \sqrt{x^2-1}} dx = \int \frac{1}{\sqrt{x^2-1}} dx = \int \frac{1}{\tan\theta} \cdot \sec\theta \tan\theta d\theta = \int \sec\theta d\theta$$

$$x = \sec\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$dx = \sec\theta \tan\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln|x + \sqrt{x^2-1}| + C$$

$$\Rightarrow \int \sec^2 x dx = x \sec^2 x - \ln|x + \sqrt{x^2-1}| + C$$

✱

43,

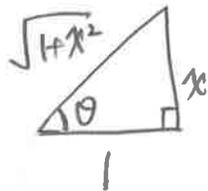
$$y = \frac{1}{1+x^2} \quad x \in [0, 1]$$

$$V = \int_0^1 \pi \left(\frac{1}{1+x^2} \right)^2 dx = \underline{\hspace{2cm}}$$

$$\int_0^1 \left(\frac{1}{1+x^2} \right)^2 dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sec^2 \theta} \right)^2 \cdot \sec^2 \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 \theta} d\theta$$

$$\begin{cases} x = \tan \theta \\ dx = \sec^2 \theta d\theta \end{cases}$$

$$\begin{cases} x=0 \rightarrow \theta=0 \\ x=1 \rightarrow \theta=\frac{\pi}{4} \end{cases}$$



$$= \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2}$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

$$\Rightarrow V = \pi \cdot \int_0^1 \left(\frac{1}{1+x^2} \right)^2 dx = \underline{\frac{\pi^2}{8} + \frac{\pi}{4}}$$

✱

10. Extra Credit - 5 points. Prove $\frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 (x^2-1)^n dx = \frac{2}{2n+1}$.

(pf) $\int_{-1}^1 (x^2-1)^n dx = \int_{\pi}^0 (-1)^n \cdot \sin^{2n} \theta \cdot (-\sin \theta) d\theta = \int_0^{\pi} (-1)^n \cdot \sin^{2n+1} \theta d\theta$
 let $x = \cos \theta$
 $dx = -\sin \theta d\theta$
 $= (-1)^n \cdot \int_0^{\pi} \sin^{2n+1} \theta d\theta$. — (1)

$$\int_0^{\pi} \sin^{2n+1} \theta d\theta = \int_0^{\pi} \sin^{2n} \theta \cdot \sin \theta d\theta = -\cos \theta \cdot \sin^{2n} \theta \Big|_0^{\pi} + \int_0^{\pi} 2n \cdot \cos^2 \theta \cdot \sin^{2n-1} \theta d\theta$$

$$= 2n \cdot \int_0^{\pi} \cos^2 \theta \cdot \sin^{2n-1} \theta d\theta$$

$$= 2n \cdot \left[\int_0^{\pi} \sin^{2n-1} \theta d\theta - \int_0^{\pi} \sin^{2n+1} \theta d\theta \right]$$

$$\Rightarrow (1+2n) \cdot \int_0^{\pi} \sin^{2n+1} \theta d\theta = 2n \cdot \int_0^{\pi} \sin^{2n-1} \theta d\theta$$

$$\Rightarrow \int_0^{\pi} \sin^{2n+1} \theta d\theta = \frac{2n}{1+2n} \int_0^{\pi} \sin^{2n-1} \theta d\theta$$

Now, $\frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 (x^2-1)^n dx = \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2} \cdot (-1)^n \cdot \int_0^{\pi} \sin^{2n+1} \theta d\theta$

$$= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{2n}{2n+1} \cdot \int_0^{\pi} \sin^{2n+1} \theta d\theta$$

$$= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \int_0^{\pi} \sin^{2n-3} \theta d\theta$$

$$\begin{aligned}
\Rightarrow \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2} \cdot \int_{-1}^1 (x^2-1)^n dx &= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \int_0^\pi \sin^{2n-3} \theta d\theta \\
&= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \int_0^\pi \sin \theta d\theta \\
&= \frac{(2n)!}{2^{2n} \cdot (n!)^2} \cdot \frac{2^n \cdot n!}{1} \cdot \frac{1}{(2n+1)(2n-1) \cdots 5 \cdot 3} \cdot \left[-\cos \theta \Big|_0^\pi \right] \\
&= \frac{(2n)!}{2^n \cdot n!} \cdot \frac{(2n) \cdot (2n-2) \cdots 4 \cdot 2 \cdot 1}{(2n+1)(2n) \cdot (2n-1)(2n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 2 \\
&= \frac{(2n)!}{2^n \cdot n!} \cdot \frac{2^n \cdot n!}{1} \cdot \frac{1}{(2n+1)!} \cdot 2 \\
&= \frac{(2n)!}{(2n+1)!} \cdot 2 \\
&= \frac{2}{2n+1}
\end{aligned}$$

