$u = \tan\frac{x}{2}$

$du = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}(1+u^2)\,dx$

$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{1-u^2}{1+u^2}$

$\sin x = \frac{2u}{1+u^2}$

1. $\int \frac{1-\cos x}{1+\sin x}\,dx$

$= \int \frac{1-\frac{1-u^2}{1+u^2}}{1+\frac{2u}{1+u^2}} \times \frac{2}{1+u^2}\,du$

$= \int \left[\frac{2u}{1+u^2} + \frac{-2}{1+u} + \frac{2}{(1+u)^2}\right] du$

$= \ln|u^2+1| - 2\ln|u+1| + \frac{2}{u+1} + C$

$= \ln\left|\frac{1}{1+\sin x}\right| - \frac{2}{\tan\frac{x}{2}+1} + C$ //

$\int \frac{1-\cos x}{1+\sin x}\,dx$

$= \int \frac{1-\cos x}{1+\sin x} \times \left(\frac{1-\sin x}{1-\sin x}\right) dx$

$= \int \frac{1-\cos x-\sin x+\sin x\cos x}{1-\sin^2 x}\,dx$

$= \int \sec^2 x\,dx - \int \sec x\,dx - \int \tan x\sec x\,dx - \int \frac{(-\sin x)}{\cos x}\,dx$

$= \tan x - \sec x - \int \left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)\sec x\,dx - \int \frac{(-\sin x)}{\cos x}\,dx$

$= \tan x - \sec x - \log|\tan x + \sec x| - \log|\cos x| + C$ //

$\tan x - \sec x = \frac{-2}{\tan\frac{x}{2}+1}$

$\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} - \frac{1+u^2}{1-u^2}$

$= \frac{2u-1-u^2}{1-u^2} = \frac{+(u-1)^2}{-(1+u)^2} = \frac{u-1}{u+1}$

$\frac{u-1}{u+1} = \frac{-2}{u+1}$

2、 $\lambda \in (0,1)$ $t>0$

$h(t) = t^{\lambda} - \lambda t$

$h'(t) = \lambda t^{\lambda-1} - \lambda = \lambda(t^{\lambda-1}-1)$

$h''(t) = \lambda(\lambda-1)t^{\lambda-2} < 0$ $(\because \lambda-1<0)$

令 $h'(t)=0 \Rightarrow t=1$

since $h''(t)<0 \Rightarrow h(1)$ is maxium

$h(1) = 1-\lambda \Rightarrow t^{\lambda} - \lambda t \le 1-\lambda$

<u>並且</u> $t^{\lambda} - \lambda t = 1-\lambda t \Leftrightarrow t=1$ (*)

$p, q \in \mathbb{R}^+$ 滿足

$\frac{1}{p}+\frac{1}{q}=1$

pf: if $u, v \ge 0$

then, $uv \le \frac{u^p}{p}+\frac{v^q}{q}$

(i) If $v=0$

$\Leftrightarrow 0 \le \frac{u^p}{p}$

(ii) $u=0 \Leftrightarrow$ "="

(iii)

If $v \ne 0$ i.e. $v>0$

$uv < \frac{u^p}{p}+\frac{v^q}{q}$

$\Leftrightarrow uv^{1-q} \le \frac{u^p}{v^q p}+\frac{1}{q}$

$\Leftrightarrow uv^{1-q} - \frac{u^p}{v^q p} \le 1-\frac{1}{p}$

take $\frac{1}{p}=\lambda$, $u^p \cdot v^{-q} = t$

$t^{\lambda} = (u^p \cdot v^{-q})^{\frac{1}{p}}$

$= uv^{\frac{1}{p}\cdot q}$

$= uv^{(1-\frac{1}{q})\cdot q}$

$= uv^{1-q} \Rightarrow t^{\lambda} - \lambda t \le 1-\lambda$

②

claim $u^p = v^q \Leftrightarrow t=1$

pf "$\Rightarrow$" suppose $u^p = v^q$, then $t = u^p \cdot v^{-q} = \frac{u^p}{v^q} = 1$

"$\Leftarrow$" $u^p \ne v^q$, $t = \frac{u^p}{v^q} \ne 1$

$\because$ "$u^p = v^q$" $\Leftrightarrow t=1$. by (*) "$u^p = v^q$" $\Leftrightarrow uv = \frac{u^p}{p}+\frac{v^q}{q}$

from (a), $uv \le \frac{u^p}{p}+\frac{v^q}{q}$

$f \ge 0, g \ge 0$ & $f, g \in R(x)$

$f \cdot g \le \frac{f^p}{p}+\frac{g^q}{q}$

$\int_a^b f \cdot g\,dx \le \int_a^b \frac{f^p}{p}dx + \int_a^b \frac{g^q}{q}dx$

$\frac{1}{p}+\frac{1}{q}$

$\int_a^b fg\,dx \le \frac{1}{p}+\frac{1}{q}=1$ //

3. $\int \frac{1}{\sinh(x)+\cosh(x)}\,dx$

$= \int \frac{1}{[2\sinh(\frac{x}{2})\cosh(\frac{x}{2})]+[\cosh^2(\frac{x}{2})+\sinh^2(\frac{x}{2})]}\,dx$

$= \int \frac{1}{[\sinh(\frac{x}{2})+\cosh(\frac{x}{2})]^2}\,dx$ (同除 $\cosh^2(\frac{x}{2})$)

$= \int \frac{\operatorname{sech}^2(\frac{x}{2})}{[\tanh(\frac{x}{2})+1]^2}\,dx$ $\left(\begin{array}{l}\text{let } u=\tanh(\frac{x}{2})\\ du=\frac{1}{2}\operatorname{sech}^2(\frac{x}{2})dx\end{array}\right)$

$= \int \frac{2}{(u+1)^2}\,du$ $\left(\begin{array}{l}\text{let } V=u+1\\ \Rightarrow dV=du\end{array}\right)$

$= \int \frac{2}{V^2}\,dV = \frac{-2}{V}+C = \frac{-2}{u+1}+C$

$= \frac{-2}{\tanh(\frac{x}{2})+1}+C$

$\sinh x = \frac{e^x-e^{-x}}{2}$

$\cosh x = \frac{e^x+e^{-x}}{2}$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

4. Pf $\|fg\|_1 \le \|f\|_p \, \|g\|_q$

$$\int_a^b f(x)\, g(x)\, dx \le \left(\int_a^b f^p(x)\,dx\right)^{1/p} \left(\int_a^b g^q(x)\,dx\right)^{1/q}$$

(b) "=" holds iff $F^p = G^q$

Let $F(s) = \dfrac{f(s)}{\left(\int_a^b f^p(x)\,dx\right)^{1/p}}$ $\quad G(s) = \dfrac{g(s)}{\left(\int_a^b g^q(x)\,dx\right)^{1/q}}$

then $\left(\dfrac{f(s)}{\left(\int_a^b f^p\,dx\right)^{1/p}}\right)^p = \left(\dfrac{g(s)}{\left(\int_a^b g^q\,dx\right)^{1/q}}\right)^q$

Since $\int_a^b F^p(s)\,ds = \int_a^b \dfrac{f^p(s)\,ds}{\left(\int_a^b f^p(x)\,dx\right)} = \dfrac{1}{\int_a^b f^p(x)\,dx} \int_a^b f^p(s)\,ds = 1$

$\int_a^b G^q(s)\,ds = 1$

$\Rightarrow f^p(s)\left(\int_a^b g^q\,dx\right) = g^q(s)\left(\int_a^b f^p\,dx\right)$

$\Rightarrow f^p(s) = \dfrac{\left(\int_a^b f^p\,dx\right)}{\left(\int_a^b g^q\,dx\right)}\, g^q(s) \Rightarrow f^p = c\,g^q$ for some $c \ne 0$,

$$\int_a^b \frac{f(s)}{\left(\int_a^b f^p(x)\,dx\right)^{1/p}} \frac{g(s)}{\left(\int_a^b g^q(x)\,dx\right)^{1/q}}\,ds \le 1 \iff \int_a^b f(x)g(x)\,dx \le \left(\int_a^b f^p(x)\,dx\right)^{1/p} \left(\int_a^b g^q(x)\,dx\right)^{1/q}$$

5. $u = x^k \quad dv = (1-x)^{n-k}\,dx$

$du = k\,x^{k-1}\,dx \quad v = \frac{-1}{n-k+1}(1-x)^{n-k+1}$

$$\boxed{\binom{n}{k} = \left[(n+1)\int_0^1 x^k (1-x)^{n-k}\,dx\right]^{-1}}$$

$$\int_0^1 x^k (1-x)^{n-k}\,dx$$

$$= x^k (1-x)^{n-k+1}\frac{-1}{n-k+1}\Big|_0^1 - \int_0^1 k\,x^{k-1}(1-x)^{n-k+1}\cdot\frac{-1}{n-k+1}\,dx = -\int_0^1 k\cdot x^{k-1}\frac{-1}{n-k+1}(1-x)^{n-k+1}\,dx$$

$(n \geq k)$

$$= \frac{k}{n-k+1}\int_0^1 x^{k-1}(1-x)^{n-k+1}\,dx$$

$$= \boxed{\frac{k}{n-k+1}\times\frac{k-1}{n-k+2}\times\cdots\times\frac{k-k+1}{n-k+k}}\int_0^1 x^0(1-x)^n\,dx$$

$$\boxed{\frac{k!}{\frac{n!}{(n-k)!}}}$$

$$= \binom{n}{k}^{-1}\cdot\left[\frac{-1}{n+1}(1-x)^{n+1}\right]_0^1 = \frac{1}{n+1}\cdot\binom{n}{k}^{-1}$$

$$\therefore (n+1)\int_0^1 x^k(1-x)^{n-k}\,dx = \binom{n}{k}^{-1}$$

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傳訊息

6. $\begin{cases} f(a)=f(b)=0 \\ \int_a^b f^2(x)\,dx=1 \end{cases}$

Prove $\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$

<pf>. let $u=x$, $dv=f(x)f'(x)dx$

$\Rightarrow du=dx$, $v=\frac{1}{2}f^2(x)$

$\int_a^b x f(x) f'(x)\,dx = \frac{1}{2}x f^2(x)\Big|_a^b - \frac{1}{2}\int_a^b f^2(x)\,dx$

$= 0 - \frac{1}{2} = -\frac{1}{2}$

(2) $\int_a^b (f'(x))^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx \geq \frac{1}{4}$

$\int_a^b (f'(x))^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx \geq \left(\int_a^b f(x)\,x f'(x)\,dx\right)^2$

$= \left(-\frac{1}{2}\right)^2$

$= \frac{1}{4}$