

Hw3

1. Salas §8-6 * 29.

$$\int \frac{1-\cos x}{1+\sin x} dx$$

<sol> $1-\cos x = 1 - (1-2\sin^2 \frac{x}{2}) = 2\sin^2 \frac{x}{2}$ and $1+\sin x = (\sin \frac{x}{2} + \cos \frac{x}{2})^2$

$$\frac{1-\cos x}{1+\sin x} = \frac{2\sin^2 \frac{x}{2}}{(\sin \frac{x}{2} + \cos \frac{x}{2})^2} = 2 \cdot \left(\frac{\sin \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} \right)^2 = 2 \cdot \left(\frac{\tan \frac{x}{2}}{1+\tan \frac{x}{2}} \right)^2$$

Let $u = \tan \frac{x}{2}$, $\frac{x}{2} = \tan^{-1}(u)$, $dx = \frac{2}{1+u^2} du$, then

$$\int \frac{1-\cos x}{1+\sin x} dx = \int 2 \cdot \left(\frac{\tan \frac{x}{2}}{1+\tan \frac{x}{2}} \right)^2 dx = \int 2 \cdot \left(\frac{u}{1+u} \right)^2 \cdot \frac{2}{1+u^2} du$$

$$= \int \frac{4u^2}{(1+u^2)(1+u)^2} du$$

$$= \int \frac{2u}{1+u^2} - \frac{2}{u+1} + \frac{2}{(u+1)^2} du$$

$$\left(\begin{array}{l} \text{Let } \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C+D}{1+u^2} = \frac{4u^2}{(1+u^2)(1+u)^2} \\ \Rightarrow \begin{cases} A+C=0 \\ A+B+2C+D=4 \\ A+D+B=0 \end{cases}, \quad A+C+2D=0, \Rightarrow A=-2, B=2, C=2, D=0 \end{array} \right)$$

$$\Rightarrow \int \frac{1-\cos x}{1+\sin x} dx = \ln(1+u^2) - 2\ln|u+1| - \frac{2}{u+1} + C$$

$$= \ln\left(\frac{1+u^2}{(1+u)^2}\right) - \frac{2}{u+1} + C$$

$$= \ln \left(\frac{1+u^2}{(1+u)^2} \right) - \frac{2}{u+1} + C, \quad u = \tan \frac{x}{2}$$

$$= \ln \left(\frac{\sec^2 \frac{x}{2}}{(1+\tan \frac{x}{2})^2} \right) - \frac{2}{1+\tan \frac{x}{2}} + C$$

$$= \ln \left(\frac{1}{(\sin \frac{x}{2} + \cos \frac{x}{2})^2} \right) - \frac{2}{1+\tan \frac{x}{2}} + C$$

$$= 2 \ln \left| \frac{1}{\sin \frac{x}{2} + \cos \frac{x}{2}} \right| - \frac{2}{1+\tan \frac{x}{2}} + C$$

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2. (a)

$h: \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

Given $\lambda \in (0, 1)$, prove that $h(t) = t^\lambda - \lambda t$ has absolute maximum at $t=1$.

What is the maximum value?

(b) With part (a), do Rudin Ch 6, *10 (a), (b).

*10. (a) If $u \geq 0$ and $v \geq 0$, then $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$, $p, q \in \mathbb{R}$.

Equality holds if and only if $u^p = v^q$.

(b) If $f \in R([a, b])$, $g \in R([a, b])$, $f \geq 0$, $g \geq 0$, and $\int_a^b f^p dx = 1 = \int_a^b g^q dx$,

then $\int_a^b fg dx \leq 1$. (Here, change all α to x , i.e., we deal with the usual Riemannian integrals here.)

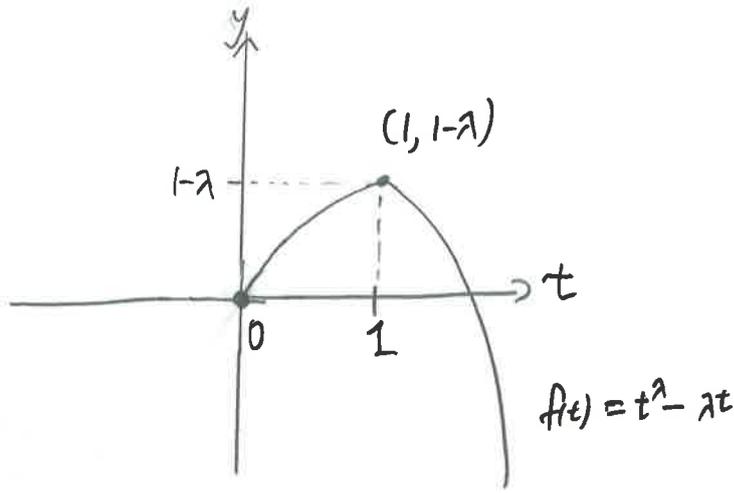
(a) <pf>

Given $0 < \lambda < 1$.

$$f'(t) = \lambda t^{\lambda-1} - \lambda = 0 \Rightarrow t^{\lambda-1} = 1 \Rightarrow t^{1-\lambda} = 1 \Rightarrow t = 1.$$

$$f''(t) = \lambda(\lambda-1) \cdot t^{\lambda-2} < 0 \quad \forall t > 0.$$

t	0	1
$f'(t)$	+	-
$f''(t)$	-	-



So $f(1) = 1 - \lambda$ is the maximum value. ▣

(b)

(pf)

10. (a) ① If $u=0$ or $v=0$, then $uv=0 \Rightarrow uv \leq \frac{u^p}{p} + \frac{v^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

② if $u > 0$ and $v > 0$, then we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q} \Leftrightarrow uv^{1-q} \leq \frac{1}{p} \cdot \frac{u^p}{v^q} + \frac{1}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$uv^{1-q} = \frac{u}{v^{q-1}} = \frac{(u^p)^{\frac{1}{p}}}{(v^q)^{1-\frac{1}{q}}} = \frac{(u^p)^{\frac{1}{p}}}{(v^q)^{\frac{1}{q}}} = \left(\frac{u^p}{v^q}\right)^{\frac{1}{p}}$$

$$\Leftrightarrow \left(\frac{u^p}{v^q}\right)^{\frac{1}{p}} \leq \frac{1}{p} \cdot \left(\frac{u^p}{v^q}\right) + 1 - \frac{1}{p}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Let $f(t) = t^{\frac{1}{p}} - \frac{1}{p}t$ be defined on $\{t \geq 0\}$, $0 < \frac{1}{p} < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Since (a), then $f(t) \leq 1 - \frac{1}{p} \quad \forall t \geq 0$

Choose $t = \frac{u^p}{v^q} > 0$, then $f\left(\frac{u^p}{v^q}\right) = \left(\frac{u^p}{v^q}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{u^p}{v^q}\right) \leq 1 - \frac{1}{p}$

$$\Leftrightarrow \left(\frac{u^p}{v^q}\right)^{\frac{1}{p}} \leq \frac{1}{p}\left(\frac{u^p}{v^q}\right) + 1 - \frac{1}{p}$$

$$\Leftrightarrow uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds $\Leftrightarrow \frac{u^p}{v^q} = 1 \Leftrightarrow u^p = v^q$.



(b)

$\langle pf \rangle$

$$\text{Let } \frac{1}{p} + \frac{1}{q} = 1.$$

$$\text{Since } uv \leq \frac{u^p}{p} + \frac{v^q}{q}, \quad u \geq 0, v \geq 0,$$

$$\text{then } f \cdot g \leq \frac{f^p}{p} + \frac{g^q}{q}, \quad f \geq 0, g \geq 0.$$

$$\Rightarrow \int_a^b fg dx \leq \int_a^b \frac{1}{p} f^p dx + \int_a^b \frac{1}{q} g^q dx$$

$$= \frac{1}{p} \int_a^b f^p dx + \frac{1}{q} \int_a^b g^q dx = \frac{1}{p} + \frac{1}{q} = 1.$$



3. Salas §8-6 * 43

$$\int \frac{1}{\sinh x + \cosh x} dx = \underline{\hspace{2cm}}$$

<sol>

$$\sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2} \quad \text{and} \quad \cosh x = \cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}$$

$$\frac{1}{\sinh x + \cosh x} = \frac{1}{(\sinh \frac{x}{2} + \cosh \frac{x}{2})^2} = \frac{1}{(1 + \tanh \frac{x}{2})^2 \cdot \cosh^2 \frac{x}{2}}$$

$$\text{Let } u = \tanh \frac{x}{2}, \quad du = \frac{1}{\cosh^2 \frac{x}{2}} \cdot \frac{1}{2} dx, \quad \text{then}$$

$$\int \frac{1}{\sinh x + \cosh x} dx = \int \frac{1}{(1 + \tanh \frac{x}{2})^2 \cdot \cosh^2 \frac{x}{2}} dx = \int \frac{2}{(1+u)^2} du$$

$$= \frac{-2}{u+1} + C$$

$$= \frac{-2}{1 + \tanh \frac{x}{2}} + C$$

*

4. Continue from Problem 2, prove

(a) for any f and g are real valued functions on $\mathcal{R}([a, b])$, we have

$$\left| \int_a^b fg dx \right| \leq \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q dx \right\}^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ p > 0, q > 0, p, q \in \mathbb{R}.$$

This is Hölder's inequality. When $p=q=2$ it is usually called the Schwarz inequality. (Note: Thm 1.35)

(b) The equality hold if and only if $|f|^p = c \cdot |g|^q$ for some $c \neq 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

(a)

<pf>

Now, $\left| \int_a^b fg dx \right| \leq \int_a^b |f| \cdot |g| dx$. Give $p > 0, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

① If $|f|=0$ or $|g|=0$, then $\int_a^b |f| \cdot |g| dx = 0 \Rightarrow \int_a^b fg dx = 0$

So $\left| \int_a^b fg dx \right| \leq \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q dx \right\}^{\frac{1}{q}}$ holds.

② If $|f| > 0$ and $|g| > 0$, we let $\|f\|_p = \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}}$,

$\|g\|_q = \left\{ \int_a^b |g|^q dx \right\}^{\frac{1}{q}}$, then $\|f\|_p > 0$ and $\|g\|_q > 0$.

$$\text{Let } \tilde{f} = \frac{|f|}{\|f\|_p} \text{ and } \tilde{g} = \frac{|g|}{\|g\|_q} \Rightarrow \tilde{f} > 0 \text{ and } \tilde{g} > 0.$$

$$\int_a^b \tilde{f}^p dx = \int_a^b \frac{|f|^p}{\|f\|_p^p} dx = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f|^p dx = 1$$

$$\int_a^b \tilde{g}^q dx = \int_a^b \frac{|g|^q}{\|g\|_q^q} dx = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g|^q dx = 1$$

Since \times 6(b), then we have

$$\int_a^b \tilde{f} \tilde{g} dx \leq 1$$

$$\Rightarrow \int_a^b \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q} dx \leq 1$$

$$\Rightarrow \int_a^b |f| \cdot |g| dx \leq \|f\|_p \cdot \|g\|_q = \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q dx \right\}^{\frac{1}{q}}$$

$$\Rightarrow \left| \int_a^b fg dx \right| \leq \int_a^b |f| \cdot |g| dx \leq \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q dx \right\}^{\frac{1}{q}}$$

(b)

$$\text{Equality holds } \Leftrightarrow \tilde{f}^p = \tilde{g}^q \Leftrightarrow \frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$$

$$\Leftrightarrow |f|^p = \frac{\|f\|_p^p}{\|g\|_q^q} \cdot |g|^q$$

$$\Leftrightarrow |f|^p = \beta \cdot |g|^q, \text{ where } \beta = \frac{\|f\|_p^p}{\|g\|_q^q} > 0.$$

5. Prove that

$$\binom{n}{k} = \left[(n+1) \int_0^1 x^k (1-x)^{n-k} dx \right]^{-1}$$

< pfs

Given $n, k \in \mathbb{N}$ and $k \leq n$.

$$\Rightarrow \int_0^1 x^k (1-x)^{n-k} dx = (1-x)^{n-k} \cdot \frac{1}{k+1} x^{k+1} \Big|_0^1 - \frac{n-k}{k+1} \cdot (-1) \cdot \int_0^1 x^{k+1} (1-x)^{n-k-1} dx$$

$$\left(\begin{array}{l} u = (1-x)^{n-k} \\ du = (n-k) \cdot (1-x)^{n-k-1} \cdot (-1) dx \end{array} \right), \quad \left(\begin{array}{l} dv = x^k dx \\ v = \frac{1}{k+1} x^{k+1} \end{array} \right)$$

$$\Rightarrow \int_0^1 x^k (1-x)^{n-k} dx = \frac{n-k}{k+1} \int_0^1 x^{k+1} (1-x)^{n-k-1} dx$$

$$= \frac{n-k}{k+1} \cdot \frac{n-k-1}{k+2} \int_0^1 x^{k+2} (1-x)^{n-k-2} dx$$

$$= \frac{n-k}{k+1} \cdot \frac{n-k-1}{k+2} \cdots \frac{1}{n} \int_0^1 x^n dx$$

$$= \frac{n-k}{k+1} \cdot \frac{n-k-1}{k+2} \cdots \frac{1}{n} \cdot \frac{1}{n+1}$$

$$\Rightarrow (n+1) \cdot \int_0^1 x^k (1-x)^{n-k} dx = \frac{n-k}{k+1} \cdot \frac{n-k-1}{k+2} \cdots \frac{1}{n}$$

$$\Rightarrow \left[(n+1) \cdot \int_0^1 x^k (1-x)^{n-k} dx \right]^{-1} = \frac{n(n-1) \cdots (k+2)(k+1)}{(n-k)(n-k-1) \cdots 1} \cdot \frac{k!}{k!}$$

$$= \frac{n!}{(n-k)! k!} = \binom{n}{k}$$



6. Rudin Ch 6, #15 (Hint: Use Problem 2.4) For the strict inequality $>$, just prove \geq .

Suppose f is a real, continuously differentiable function on $[a, b]$,

$$f(a) = f(b) = 0, \text{ and } \int_a^b f(x)^2 dx = 1.$$

Prove that $\int_a^b x f(x) f'(x) dx = \frac{-1}{2}$ and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f(x)^2 dx \geq \frac{1}{4}$$

$\langle \text{pt} \rangle$

$$\textcircled{1} \int_a^b x f(x) f'(x) dx = \frac{1}{2} x \cdot f(x)^2 \Big|_a^b - \int_a^b \frac{1}{2} f(x)^2 dx = \frac{-1}{2} \int_a^b f(x)^2 dx = \frac{-1}{2}$$

$$u = x, \quad dv = f(x) f'(x) dx$$

$$du = dx, \quad v = \frac{1}{2} f(x)^2$$

$\textcircled{2}$ By Hölder's inequality, then choose $p = q = 2$, we have

$$\left| \int_a^b x f(x) f'(x) dx \right| \leq \left(\int_a^b x^2 f(x)^2 dx \right)^{\frac{1}{2}} \cdot \left(\int_a^b (f'(x))^2 dx \right)^{\frac{1}{2}}$$

$$\Rightarrow \left(\int_a^b x f(x) f'(x) dx \right)^2 \leq \left(\int_a^b x^2 f(x)^2 dx \right) \cdot \left(\int_a^b (f'(x))^2 dx \right)$$

$$\Rightarrow \frac{1}{4} \leq \left(\int_a^b x^2 f(x)^2 dx \right) \cdot \left(\int_a^b (f'(x))^2 dx \right)$$



7. Salas § 8-5 * 2, 6, 15, 22, 28.

2. Let $\frac{x^2}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$.

$$\Rightarrow A(x^2+4x+5) + (x-1)(Bx+C) = (A+B)x^2 + (4A-B+C)x - C + 5A = x^2$$

$$\begin{cases} A+B=1 \\ 4A-B+C=0 \\ 5A-C=0 \end{cases} \Rightarrow A = \frac{1}{10}, B = \frac{9}{10}, C = \frac{1}{2}$$

$$\Rightarrow \frac{x^2}{(x-1)(x^2+4x+5)} = \frac{\frac{1}{10}}{x-1} + \frac{\frac{9}{10}x + \frac{1}{2}}{x^2+4x+5} \quad \#$$

6. Let $\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$.

$$\Rightarrow x^3+x^2+x+2 = (A+C)x^3 + (B+C)x^2 + (2A+C)x + (2B+D)$$

$$\begin{cases} A+C=1 \\ B+C=1 \\ 2A+C=1 \\ 2B+D=2 \end{cases} \Rightarrow A=0, B=1, C=1, D=0$$

$$\Rightarrow \frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{1}{x^2+1} + \frac{x}{x^2+2} \quad \#$$

$$15. \int \frac{x+3}{x^2-3x+2} dx = \underline{\hspace{2cm}}$$

<sol>

$$\text{Let } \frac{x+3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} \Rightarrow (A+B)x + (-2A-B) = x+3$$

$$\begin{cases} A+B=1 \\ -2A-B=3 \end{cases} \Rightarrow A=-4 \quad B=5$$

$$\Rightarrow \frac{x+3}{x^2-3x+2} = \frac{-4}{x-1} + \frac{5}{x-2} \Rightarrow \int \frac{x+3}{x^2-3x+2} dx = \int \frac{-4}{x-1} dx + \int \frac{5}{x-2} dx$$

$$= -4 \ln|x-1| + 5 \ln|x-2| + C$$

$$22. \int \frac{3x^5-3x^2+x}{x^3-1} dx = \underline{\hspace{2cm}}$$

<sol>

$$\frac{3x^5-3x^2+x}{x^3-1} = 3x^2 + \frac{x}{x^3-1}, \quad \frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow x = (A+B)x^2 + (A-B+C)x + (A-C)$$

$$\begin{cases} A+B=0 \\ A-B+C=1 \\ A-C=0 \end{cases} \Rightarrow A=\frac{1}{3}, B=-\frac{1}{3}, C=\frac{1}{3} \Rightarrow \frac{x}{x^3-1} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}$$

$$\Rightarrow \int \frac{3x^5-3x^2+x}{x^3-1} dx = \int 3x^2 dx + \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{x-1}{x^2+x+1} dx$$

$$= x^3 + \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{2x+1}{x^2+x+1} \cdot \frac{1}{2} + \frac{-\frac{3}{2}}{x^2+x+1} dx$$

$$= x^3 + \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \int \frac{1}{x^2+x+1} dx$$

$$\int \frac{1}{x^2+x+1} dx = \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx = \int \frac{1}{u^2 + \frac{3}{4}} du, \text{ let } u = x + \frac{1}{2}, du = dx,$$

$$\text{let } u = \frac{\sqrt{3}}{2}s, du = \frac{\sqrt{3}}{2} ds$$

$$= \int \frac{4}{3} \cdot \frac{1}{s^2+1} \cdot \frac{\sqrt{3}}{2} ds$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{1+s^2} ds = \frac{2}{\sqrt{3}} \cdot \tan^{-1}(s) + C = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

$$\Rightarrow \int \frac{3x^5 - 3x^2 + x}{x^3 - 1} dx = x^3 + \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

$$28. \int \frac{1}{(x-1)(x^2+1)^2} dx = \underline{\hspace{2cm}}$$

$$\text{sol/} \text{ Let } \frac{1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow 1 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)$$

$$\Rightarrow 1 = A(x^4+2x^2+1) + (Bx+C)(x^3-x^2+x-1) + Dx^2+Ex-Dx-E$$

$$\Rightarrow 1 = (A+B)x^4 + (-B+C)x^3 + (2A+B-C+D)x^2 + (-B+C+E-D)x + (A-C-E)$$

$$A+B=0$$

$$-B+C=0$$

$$2A+B-C+D=0$$

$$-B+C+E-D=0$$

$$A-C-E=1$$

$$\Rightarrow A = \frac{1}{4} \quad C = \frac{1}{4} \quad E = \frac{1}{2}$$

$$B = \frac{1}{4} \quad D = \frac{1}{2}$$

$$\Rightarrow \frac{1}{(x-1)(x^2+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x - \frac{1}{2}}{(x^2+1)^2}$$

$$\therefore \int \frac{1}{x-1} dx = \frac{1}{4} \ln|x-1| + C \quad \text{--- ①}$$

$$\int \frac{\frac{1}{4}x - \frac{1}{4}}{x^2+1} dx = \frac{1}{4} \int \frac{x}{x^2+1} dx - \frac{1}{4} \int \frac{1}{x^2+1} dx$$

$$= \frac{1}{8} \int \frac{2x}{x^2+1} dx - \frac{1}{4} \tan^{-1}(x) = \frac{1}{8} \ln(x^2+1) - \frac{1}{4} \tan^{-1}(x) + C \quad (2)$$

$$\int \frac{\frac{1}{2}x - \frac{1}{2}}{(x^2+1)^2} dx = \frac{1}{2} \int \frac{x}{(x^2+1)^2} dx - \frac{1}{2} \int \frac{1}{(x^2+1)^2} dx$$

$$\int \frac{x}{(x^2+1)^2} dx = \int \frac{1}{(1+x^2)^2} d(\frac{1}{2}x^2 + \frac{1}{2}) = \frac{1}{2} \int \frac{1}{(1+x^2)^2} d(x^2+1) = \frac{1}{2} \cdot \frac{-1}{1+x^2} + C$$

$$\Rightarrow \frac{1}{2} \int \frac{x}{(1+x^2)^2} dx = \frac{1}{4} \cdot \frac{1}{1+x^2} + C \quad (3)$$

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{1}{\sec^2 \theta} \cdot \sec^2 \theta d\theta = \int \frac{1}{\sec \theta} d\theta = \int \cos \theta d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta$$

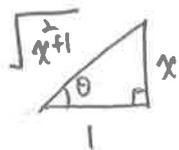
$$= \frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) + C$$

$$= \frac{1}{2} \tan^{-1}(x) + \frac{1}{4} \cdot 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2} \tan^{-1}(x) + \frac{1}{2} \cdot \frac{x}{1+x^2} + C \quad (4)$$

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$



By (1), (2), (3), (4), then

$$\int \frac{1}{(x-1)(x^2+1)} dx = \frac{1}{4} \ln|x-1| - \frac{1}{8} \ln(x^2+1) - \frac{1}{4} \tan^{-1}(x) + \frac{1}{4} \cdot \frac{1}{1+x^2} - \frac{1}{2} \left(\frac{1}{2} \tan^{-1}(x) + \frac{1}{2} \cdot \frac{x}{1+x^2} \right) + C$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{8} \ln(x^2+1) - \frac{1}{2} \tan^{-1}(x) + \frac{1}{4} \cdot \frac{1}{1+x^2} - \frac{1}{4} \cdot \frac{x}{1+x^2} + C$$

~~✗~~

8. Salas 88.6 * 2, 12

$$2_1 \int \frac{\sqrt{x}}{1+x} dx = \int \frac{u}{1+u^2} \cdot 2u du = \int \frac{2u^2}{1+u^2} du$$

$$u = \sqrt{x}, \quad u^2 = x, \quad dx = 2u du \quad = \int 2 + \frac{-2}{1+u^2} du$$

$$= 2u - 2 \tan^{-1}(u) + C$$

$$= 2\sqrt{x} - 2 \tan^{-1}(\sqrt{x}) + C$$

*

$$12_1 \int \frac{x}{\sqrt{1+x}} dx = \int \frac{u^2-1}{u} \cdot 2u du = \int 2(u^2-1) du$$

$$u = \sqrt{1+x}$$

$$u^2 = 1+x, \quad 2u du = dx$$

$$= \int 2u^2 - 2 du$$

$$= \frac{2}{3} u^3 - 2u + C$$

$$= \frac{2}{3} (1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + C$$

*

(2) Show that $g(x) = Ax^2 + Bx + C$ satisfies the condition

$$\int_a^b g(x) dx = \frac{b-a}{6} [g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)] \text{ for every interval } [a, b].$$

(pf)

$$\text{L.H.S.} = \frac{b-a}{6} [g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)]$$

$$= \frac{b-a}{6} [Aa^2 + Ba + C + 4\left(A \cdot \left(\frac{a+b}{2}\right)^2 + B \cdot \frac{a+b}{2} + C\right) + A \cdot b^2 + B \cdot b + C]$$

$$= \frac{b-a}{6} [Aa^2 + Ba + C + \underbrace{Aa^2 + 2Aab + Ab^2}_{2A(a^2 + ab + b^2)} + \underbrace{2Ba + 2Bb}_{3B(a+b)} + \underbrace{4C + Ab^2 + Bb + C}_{6C}]$$

$$= \frac{b-a}{6} [2A(a^2 + ab + b^2) + 3B(a+b) + 6C]$$

$$= \frac{1}{3}A(b^3 - a^3) + \frac{1}{2}B(b^2 - a^2) + C(b-a)$$

$$\text{R.H.S.} = \int_a^b Ax^2 + Bx + C dx = \frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx \Big|_{x=a}^{x=b}$$

$$= \frac{1}{3}A(b^3 - a^3) + \frac{1}{2}B(b^2 - a^2) + C(b-a)$$

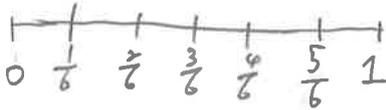
9. Salas § 8.7 * 5.9, 12

$$5. \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$$

(a) $n=6$, the trapezoidal rule (b) Simpson's rule, $n=3$

<sol>

(a)



$$f(x) = \frac{1}{1+x^2}$$

$$T_6 = \frac{1-0}{6} \times \left[\frac{f(0)+f(\frac{1}{6})}{2} + \frac{f(\frac{1}{6})+f(\frac{2}{6})}{2} + \frac{f(\frac{2}{6})+f(\frac{3}{6})}{2} + \frac{f(\frac{3}{6})+f(\frac{4}{6})}{2} + \frac{f(\frac{4}{6})+f(\frac{5}{6})}{2} + \frac{f(\frac{5}{6})+f(1)}{2} \right]$$

$$= \frac{1}{6} \times \left[1 + 2 \left(\frac{36}{37} + \frac{36}{40} + \frac{36}{45} + \frac{36}{52} + \frac{36}{61} \right) + \frac{1}{2} \right]$$

$$\approx 0.7826543044$$

$$\Rightarrow \pi \approx 0.7826543044 \times 4 = 3.1306172176$$

✘

(b)

$$f(x) = \frac{1}{1+x^2}$$

$$S_3 = \frac{1-0}{6 \times 3} \times \left[f(0) + 2 \left(f(\frac{1}{3}) + f(\frac{2}{3}) \right) + 4 \left(f(\frac{1}{6}) + f(\frac{1}{2}) + f(\frac{5}{6}) \right) + f(1) \right]$$

$$= \frac{1}{18} \left[1 + 2 \left(\frac{9}{10} + \frac{9}{13} \right) + 4 \left(\frac{36}{37} + \frac{4}{5} + \frac{36}{61} \right) + \frac{1}{2} \right]$$

$$\approx 0.7853979452$$

$$\Rightarrow \pi \approx 0.7853979452 \times 4 = 3.1415917809$$

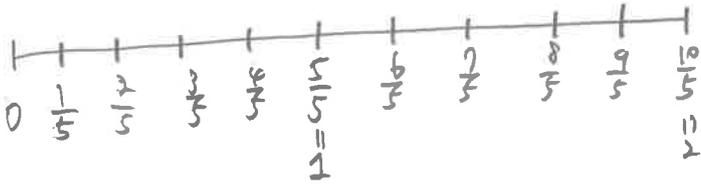
✘

9. $\int_0^2 e^{-x^2} dx = \underline{\hspace{2cm}}$

(a) the trapezoidal rule, $n=10$

(b) Simpson's rule, $n=5$

<S₁₀>
(a)



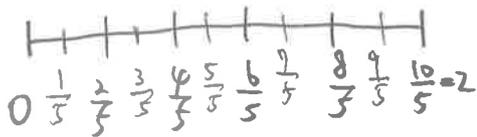
$f(x) = e^{-x^2}$

$$T_{10} = \frac{2-0}{10} \times \left[\frac{f(0) + f(\frac{1}{5})}{2} + \frac{f(\frac{1}{5}) + f(\frac{2}{5})}{2} + \frac{f(\frac{2}{5}) + f(\frac{3}{5})}{2} + \frac{f(\frac{3}{5}) + f(\frac{4}{5})}{2} + \frac{f(\frac{4}{5}) + f(1)}{2} \right. \\ \left. + \frac{f(1) + f(\frac{6}{5})}{2} + \frac{f(\frac{6}{5}) + f(\frac{7}{5})}{2} + \frac{f(\frac{7}{5}) + f(\frac{8}{5})}{2} + \frac{f(\frac{8}{5}) + f(\frac{9}{5})}{2} + \frac{f(\frac{9}{5}) + f(2)}{2} \right]$$

≈ 0.8818

*

(b)



$f(x) = e^{-x^2}$

$$S_5 = \frac{2-0}{6 \times 5} \times \left[f(0) + 2 \left(f(\frac{2}{5}) + f(\frac{4}{5}) + f(\frac{6}{5}) + f(\frac{8}{5}) \right) + 4 \left(f(\frac{1}{5}) + f(\frac{3}{5}) + f(1) \right. \right. \\ \left. \left. + f(\frac{7}{5}) + f(\frac{9}{5}) \right) + f(2) \right]$$

≈ 0.8821

*

10. Salas § 11.7 * 4, 18, 25, 30, 32, 39, 54, 58, 66, 67

$$4. \int_0^{\infty} e^{-px} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-px} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{p} - \frac{1}{p} e^{-pb} \right) = \frac{1}{p} \text{ (converges)}$$

$$\int_0^b e^{-px} dx = \left. -\frac{1}{p} e^{-px} \right|_{x=0}^{x=b} = \frac{1}{p} e^{-pb} + \frac{1}{p}$$

$$18. \int_2^{\infty} \frac{1}{x^2-1} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x^2-1} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{2} \ln \left| \frac{b-1}{b+1} \right| + \ln \sqrt{3} \right) = \ln \sqrt{3} \text{ (converges)}$$

$$\int_2^b \frac{1}{x^2-1} dx = \left. -\ln \left| \frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}} \right| \right|_{x=2}^{x=b} = -\ln \left| \frac{b}{\sqrt{b^2-1}} + \frac{1}{\sqrt{b^2-1}} \right| + \ln \sqrt{3}$$

$$= \frac{1}{2} \ln \left| \frac{b-1}{b+1} \right| + \ln \sqrt{3}$$

$$25. \int_{-3}^3 \frac{dx}{x(x+1)} = ?$$

$$\text{Now, } \int \frac{1}{x(x+1)} dx = \int \frac{1}{x} - \frac{1}{x+1} dx = \ln|x| - \ln|x+1| + C = \ln \left| \frac{x}{x+1} \right| + C$$

$$\int_0^3 \frac{1}{x(x+1)} dx = \lim_{a \rightarrow 0^+} \int_a^3 \frac{1}{x(x+1)} dx = \lim_{a \rightarrow 0^+} \left(\ln \frac{3}{4} - \ln \frac{a}{a+1} \right) = +\infty \text{ (diverges)}$$

$$\text{So } \int_{-3}^3 \frac{dx}{x(x+1)} \text{ diverges.}$$

$$30. \int_1^2 \frac{1}{x^2 - 5x + 6} dx = \int_1^2 \frac{1}{(x-2)(x-3)} dx = \lim_{b \rightarrow 2^-} \int_1^b \frac{1}{(x-2)(x-3)} dx$$

$$= \lim_{b \rightarrow 2^-} \left[\int_1^b \frac{1}{x-3} - \frac{1}{x-2} dx \right]$$

$$= \lim_{b \rightarrow 2^-} \left[\ln|x-3| \Big|_{x=1}^{x=b} - \ln|x-2| \Big|_{x=1}^{x=b} \right]$$

$$= \lim_{b \rightarrow 2^-} \left[\ln|b-3| - \ln 2 - \ln|b-2| \right]$$

$$= \lim_{b \rightarrow 2^-} \left[\ln \left| \frac{b-3}{b-2} \right| - \ln 2 \right] = +\infty \quad (\text{diverges})$$

$$So \int_1^4 \frac{1}{x^2 - 5x + 6} dx \text{ diverges.} \quad \times$$

$$32. \int_0^{\infty} \cos^2 x dx = \underline{\hspace{2cm}}$$

$$\int_0^{\infty} \cos^2 x dx = \lim_{b \rightarrow \infty} \int_0^b \cos^2 x dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1 + \cos 2x}{2} dx$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{2}x + \frac{1}{4}\sin 2x \Big|_{x=0}^{x=b} \right)$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{2}b + \frac{1}{4}\sin(2b) \right) = \underline{\hspace{2cm}} \quad (\text{diverges}) \quad \times$$

39
(sol)

$$\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = \int_0^1 \frac{1}{\sqrt{x}(1+x)} dx + \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$
$$= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)} dx + \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}(1+x)} dx$$

$$\int \frac{1}{\sqrt{x}(1+x)} dx = \int \frac{1}{u(1+u^2)} \cdot 2u du = 2 \int \frac{1}{1+u^2} du = 2 \tan^{-1}(u) + C = 2 \tan^{-1}(\sqrt{x}) + C$$

$$u = \sqrt{x}, \quad dx = 2u du$$
$$x = u^2$$

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)} dx = \lim_{a \rightarrow 0^+} \left(2 \tan^{-1}(\sqrt{x}) \Big|_{x=a}^{x=1} \right) = \lim_{a \rightarrow 0^+} \left(\frac{\pi}{2} - 2 \tan^{-1}(\sqrt{a}) \right) = \frac{\pi}{2}$$

$$\lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}(1+x)} dx = \lim_{b \rightarrow \infty} \left(2 \tan^{-1}(\sqrt{x}) \Big|_{x=1}^{x=b} \right) = \lim_{b \rightarrow \infty} \left(2 \tan^{-1}(\sqrt{b}) - \frac{\pi}{2} \right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{So } \int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\pi}$$

✱

$$54. \int_{\pi}^{\infty} \frac{\sin^2(2x)}{x^2} dx = \underline{\hspace{2cm}}$$

<Sol> $\because 0 \leq \sin^2(2x) \leq 1 \Rightarrow 0 \leq \frac{\sin^2(2x)}{x^2} \leq \frac{1}{x^2}$ as $x \geq \pi$

$$\Rightarrow 0 \leq \int_{\pi}^{\infty} \frac{\sin^2(2x)}{x^2} dx \leq \int_{\pi}^{\infty} \frac{1}{x^2} dx$$

$$\because \int_{\pi}^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_{\pi}^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{x} \Big|_{\pi}^b \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{\pi} \right) = \frac{1}{\pi} \text{ (Converges)}$$

by comparison test, then $\int_{\pi}^{\infty} \frac{\sin^2(2x)}{x^2} dx$ converges. *

58. Show that (a) $\int_{-\infty}^{\infty} \sin x dx$ diverges although (b) $\lim_{b \rightarrow \infty} \int_{-b}^b \sin x dx = 0$.

<pf>

$$(a) \because \int_0^{\infty} \sin x dx = \lim_{b \rightarrow \infty} \int_0^b \sin x dx = \lim_{b \rightarrow \infty} (-\cos x \Big|_0^b) = \lim_{b \rightarrow \infty} (1 - \cos b) \text{ does not exist}$$

$\therefore \int_{-\infty}^{\infty} \sin x dx$ diverges.

(b)

$$\lim_{b \rightarrow \infty} \int_{-b}^b \sin x dx = \lim_{b \rightarrow \infty} (-\cos x \Big|_{-b}^b) = \lim_{b \rightarrow \infty} (-\cos b + \cos(-b))$$

$$= \lim_{b \rightarrow \infty} 0$$

$$= 0.$$

~~□~~

66. Let $k > 0$. Show that

$$f(x) = \begin{cases} ke^{-kx}, & x \geq 0 \\ 0, & x < 0 \end{cases} \text{ is probability density function.}$$

It is called the exponential density function.

(p1) ①
Clearly, $f(x)$ is nonnegative.

$$\begin{aligned} \text{② } \int_{-\infty}^{\infty} f(x) dx &= \int_0^{\infty} ke^{-kx} dx = \lim_{b \rightarrow \infty} \int_0^b ke^{-kx} dx \\ &= \lim_{b \rightarrow \infty} \left(-e^{-kx} \Big|_0^b \right) \\ &= \lim_{b \rightarrow \infty} (-e^{-bk} + 1) \\ &= 1 \end{aligned}$$

By ①, ②, then $f(x)$ is called a probability density function. ■

67. The mean of a probability density function f is defined as the number

$$\mu = \int_{-\infty}^{\infty} x f(x) dx.$$

Calculate the mean for the exponential density function.

<sol>

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} kx e^{-kx} dx = \lim_{b \rightarrow \infty} \int_0^b kx e^{-kx} dx.$$

$$\begin{aligned} \int_0^b kx \cdot e^{-kx} dx &= \int_0^b x d(-e^{-kx}) \\ &= -x e^{-kx} \Big|_0^b + \int_0^b e^{-kx} dx \\ &= -b e^{-bk} + \left(-\frac{1}{k} e^{-kx} \Big|_0^b \right) \\ &= -b e^{-bk} - \frac{1}{k} e^{-bk} + \frac{1}{k} \end{aligned}$$

$$\Rightarrow \mu = \lim_{b \rightarrow \infty} \left(\frac{-b}{e^{bk}} - \frac{1}{k e^{bk}} + \frac{1}{k} \right) =$$

$$\because \lim_{b \rightarrow \infty} \frac{-b}{e^{bk}} = \left(\frac{\infty}{\infty} \right) = \lim_{b \rightarrow \infty} \frac{-1}{k e^{bk}} = 0, \quad \lim_{b \rightarrow \infty} \frac{1}{k e^{bk}} = 0$$

$$\therefore \mu = 0 - 0 + \frac{1}{k} = \frac{1}{k}$$

*

11. Show that $\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 \cdot f(x)^2 dx > \frac{1}{4}$ (see problem 6.)

<pf> Suppose $\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 \cdot f(x)^2 dx = \frac{1}{4}$.

Since the equality holds, then $|f'(x)|^2 = \beta \cdot |x f(x)|^2$ for some $\beta > 0$.

$$\Rightarrow |f'(x)| = \beta^{\frac{1}{2}} \cdot |x| \cdot |f(x)|, \quad M = \max\{|a|, |b|\}$$



$$\Rightarrow |f'(x)| \leq \beta^{\frac{1}{2}} \cdot M \cdot |f(x)|$$

Now, f is differentiable on $[a, b]$, $f(a) = f(b) = 0$, and

$|f'(x)| \leq \sqrt{\beta} \cdot M \cdot |f(x)|$ on $[a, b]$, then $f(x) = 0$ on $[a, b]$.

(see Rudin, ch 5 * 26)

$\Rightarrow \int_a^b f(x)^2 dx = 0$. (This is a contradiction.)

So $\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 \cdot f(x)^2 dx > \frac{1}{4}$.



