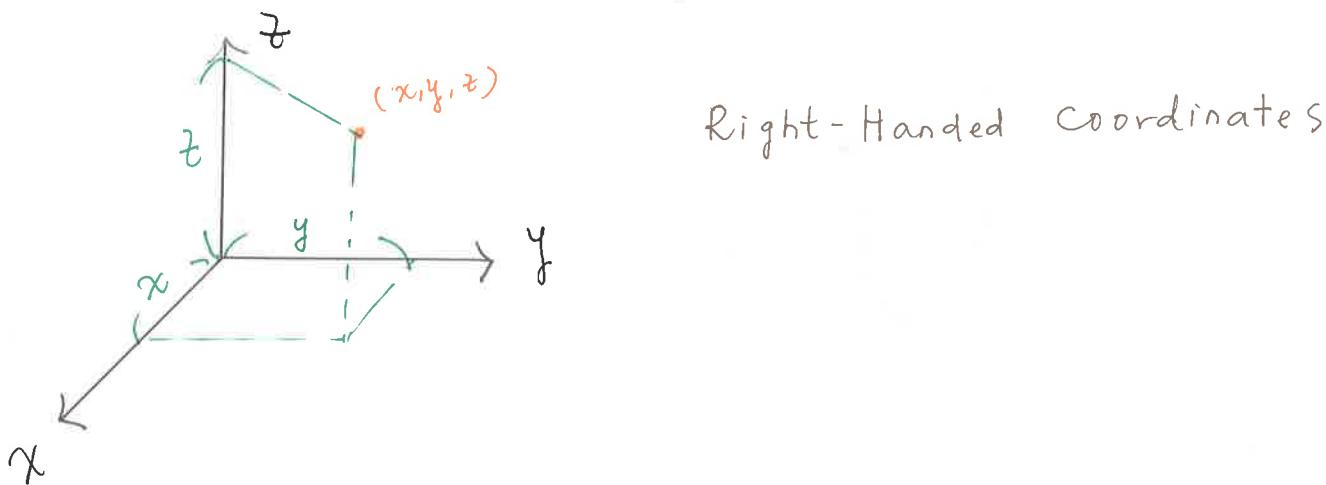


III. Vectors in \mathbb{R}^3

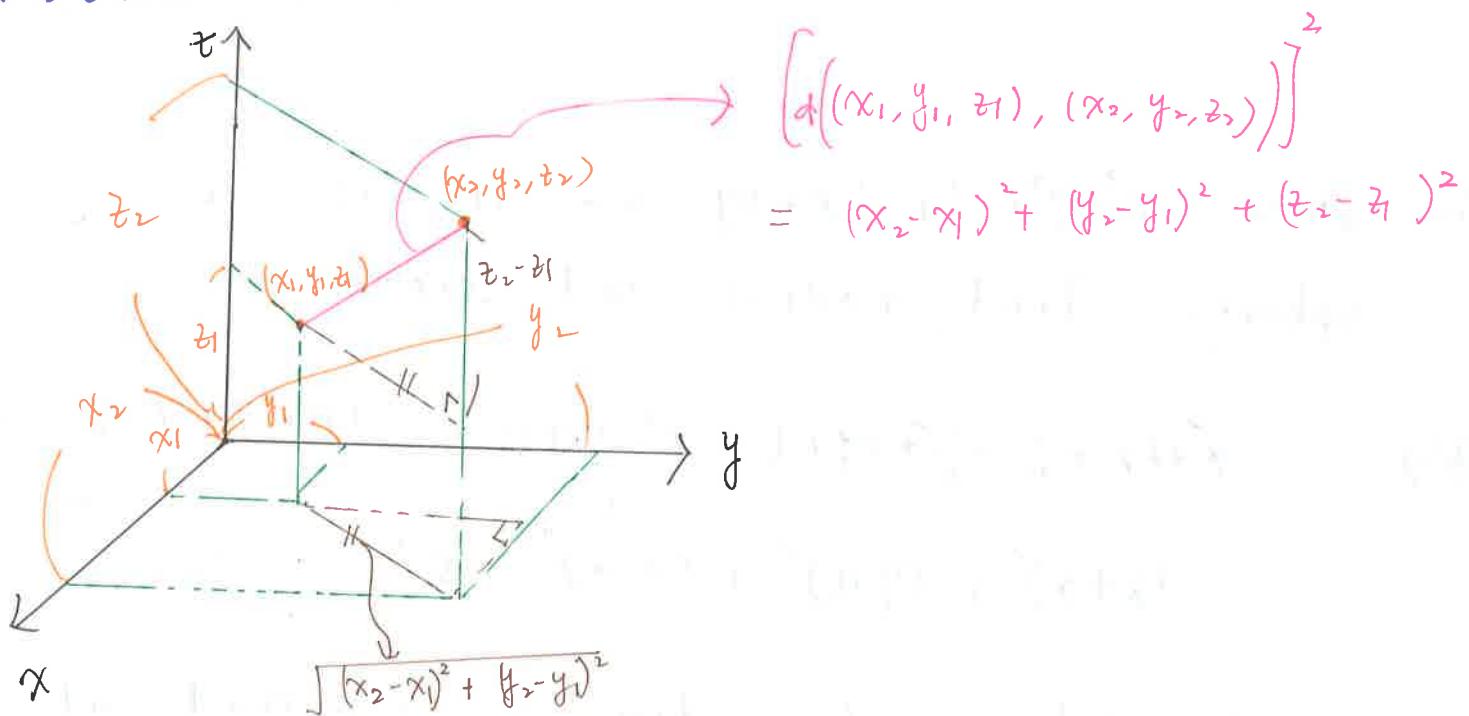
* Rectangular Space Coordinates (\mathbb{R}^3)

Defn $\mathbb{R}^3 = \left\{ \underbrace{(x, y, z)}_{\text{a point}} \mid x, y, z \in \mathbb{R} \right\}$

Graphical Representation



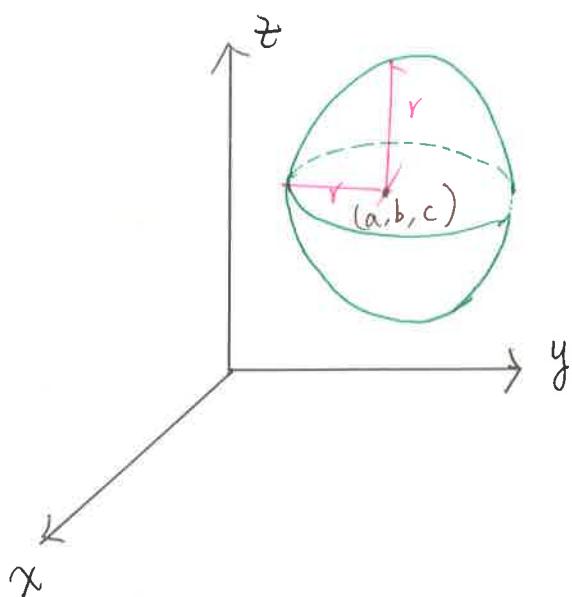
Distance Between Two Points



(2)

Defn Sphere with radius r centered at $O = (a, b, c)$ is the collection of all points with distance r to O ; ie. all points (x, y, z) s.t.

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$



e.g. show $x^2 + y^2 + z^2 + 6x + 2y - 4z = 11$ is a sphere. Find radius and center.

$$\textcircled{*} \Rightarrow x^2 + 6x + 9 + y^2 + 2y + 1 + z^2 - 4z + 4 = 11 + 9 + 1 + 4 = 25$$

$$(x+3)^2 + (y+1)^2 + (z-2)^2 = 5^2$$

\therefore sphere w/ radius 5 centered at $(-3, -1, 2)$

* Vectors in \mathbb{R}^3

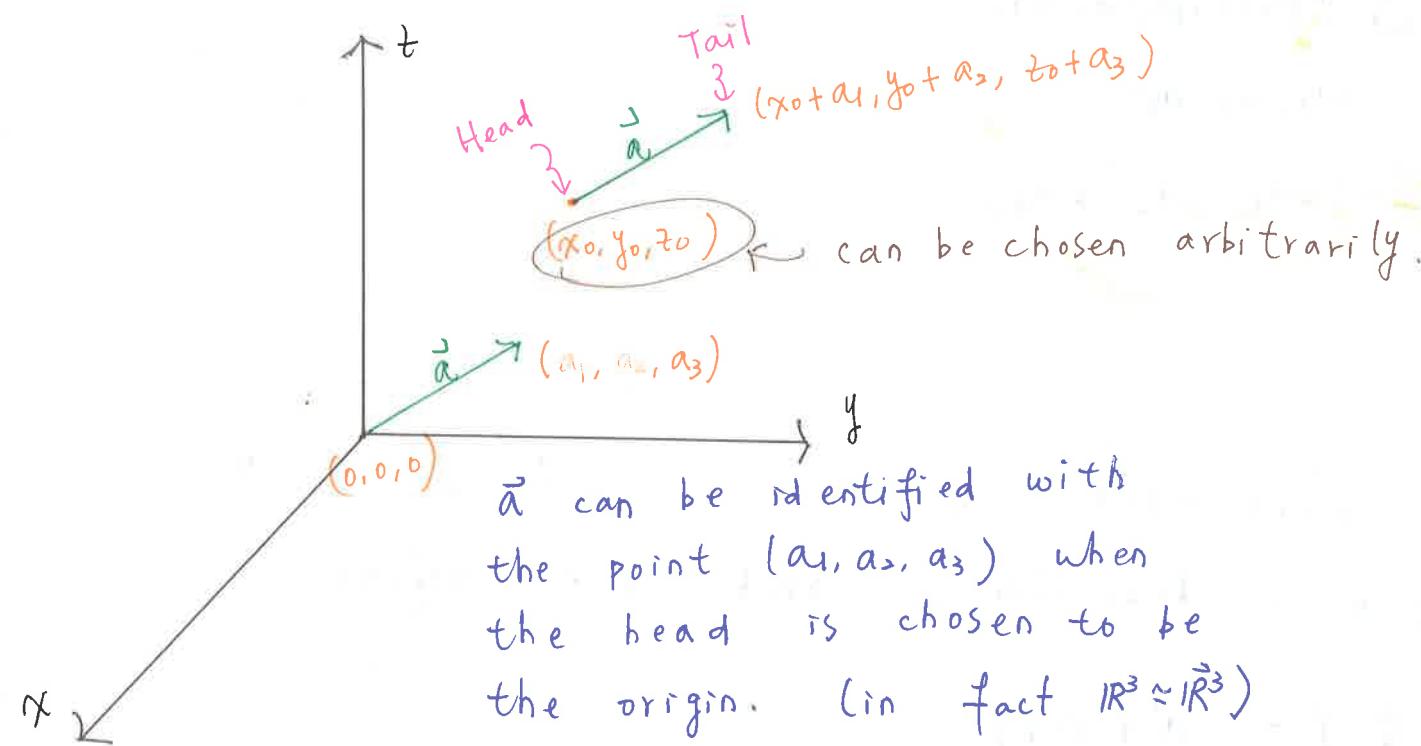
(3)

A vector is a direction and magnitude, represented by an ordered triple of real numbers.

$$\vec{a} = (a_1, a_2, a_3)$$

and we denote $\vec{\mathbb{R}}^3$ for the collection of vectors in \mathbb{R}^3 .

Graphical Representation:



* Vector Arithmetics

Algebra

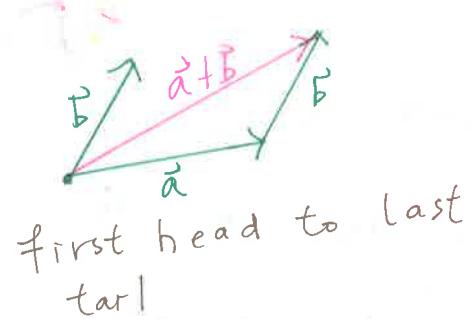
Geometry

Addition

$$\vec{a} = (a_1, a_2, a_3)$$

$$\vec{b} = (b_1, b_2, b_3)$$

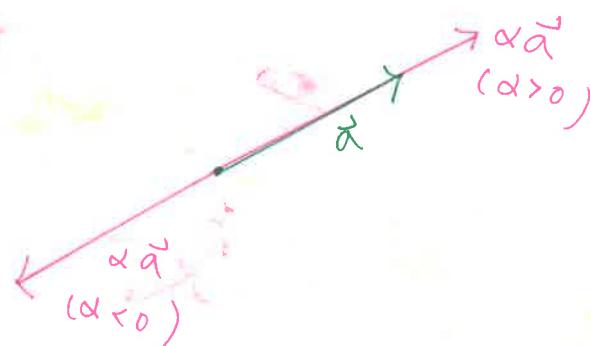
$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$



Scalar Multiplication

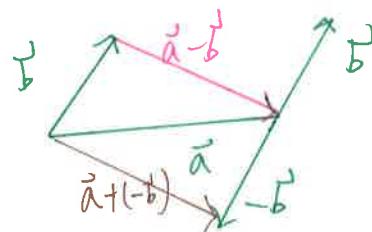
$$\vec{a} = (a_1, a_2, a_3)$$

$$\alpha \vec{a} = (\alpha a_1, \alpha a_2, \alpha a_3)$$



in particular, if $\alpha = -1$, $-\vec{a}$ is \vec{a} in opposite direction, and we define

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$



$$\text{eg} \quad \vec{a} = (1, -1, 2)$$

$$\vec{b} = (2, 3, -1)$$

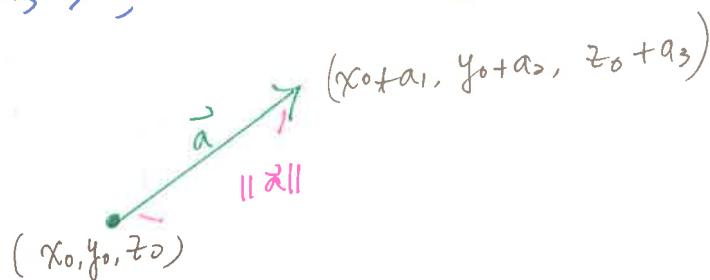
$$2\vec{a} + \vec{b} = (2, -2, 4) + (2, 3, -1)$$

$$= (4, 1, 3)$$

* More Properties

① Magnitude (Norm, Length) of \vec{a}

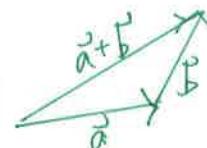
$$\vec{a} = (a_1, a_2, a_3), \quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



1a. $\|\vec{a}\| > 0$, and $= 0 \Leftrightarrow \vec{a} = 0$

1b. $\|\alpha\vec{a}\| = |\alpha| \|\vec{a}\|$

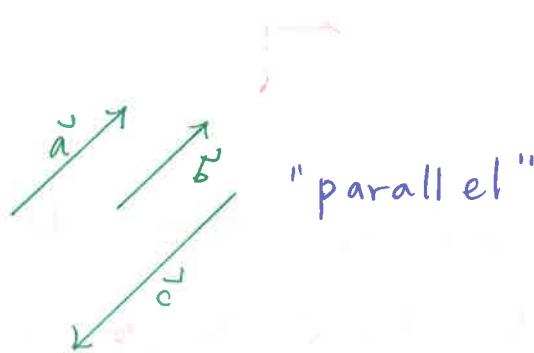
1c. $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$



② Parallel Vectors.

two vectors \vec{a}, \vec{b} are parallel, if

$$\vec{b} = \alpha \vec{a}, \quad \alpha \neq 0$$

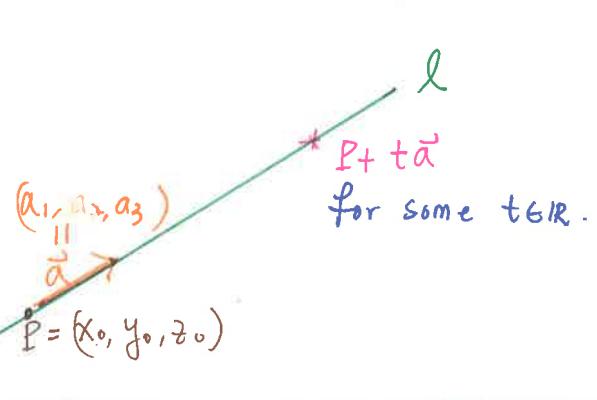


③ Parametric Equation of a Line in \mathbb{R}^3

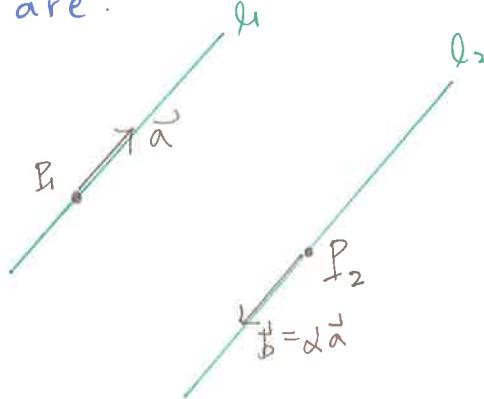
A line is determined by a point P and a direction vector \vec{a}

Every point on ℓ is of the form

$$P + t\vec{a} = (x_0 + ta_1, y_0 + ta_2, z_0 + ta_3)$$



3a. Two lines are parallel if their direction vectors are. (6)



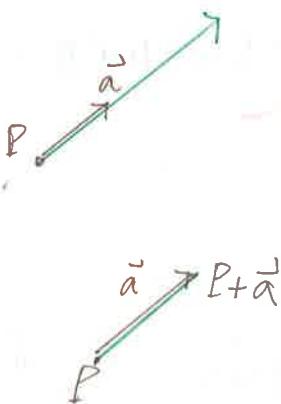
and $\vec{a} = \vec{b}$ if $P_1 = P_2$

(i.e. two parallel lines are the same line if they have one point in common)

3b. Restriction on $t \rightarrow$ subset of the line.

$$\text{eg: } l = \{P + t\vec{a} \mid t > 0\}$$

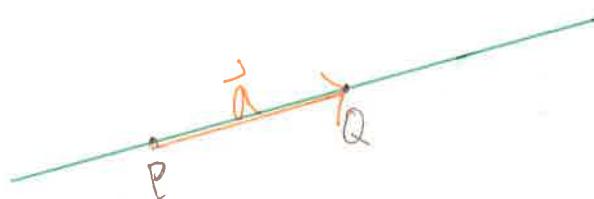
$$l = \{P + t\vec{a} \mid 0 \leq t \leq 1\}$$



3c. A line is also determined by two distinct points P, Q :

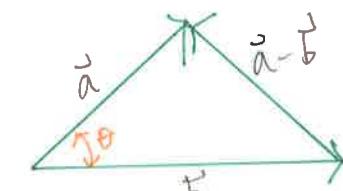
Given P, Q , let $\vec{a} = \vec{Q} - \vec{P}$

the line thru P, Q is $P + t\vec{a}$



* Dot Product

Here we develop a way to measure how parallel two vectors are ...



$$\vec{a} = (a_1, a_2, a_3)$$

$$\vec{b} = (b_1, b_2, b_3)$$

law of cosine:

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2$$

$$- (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2$$

$$= 2(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$\vec{a} \cdot \vec{b}$$

Def_{II} (Dot Product)

$$\text{We define } \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Observe : if $\vec{a}, \vec{b} \neq 0$, $\vec{a} \cdot \vec{b} = 0$ if and only if $\cos\theta = 0$, i.e. $\vec{a} \perp \vec{b}$.

\therefore if \vec{a}, \vec{b} have fixed length, $\vec{a} \cdot \vec{b}$ is max.

when $\cos\theta = 1$, i.e. when they are parallel.

Some Basic Properties

$$\textcircled{1} \quad \vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$

$$\textcircled{2} \quad \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \quad (\text{dot product is commutative})$$

$$\textcircled{3} \quad \alpha \vec{a} \cdot \beta \vec{b} = \alpha \beta \vec{a} \cdot \vec{b} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{dot product is}$$

$$\textcircled{4} \quad \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{bilinear.}$$

$$\text{eg} \quad \|\vec{a}\| = 1, \quad \|\vec{b}\| = 3, \quad \|\vec{c}\| = 4, \quad \vec{a} \cdot \vec{b} = 0, \quad \vec{a} \cdot \vec{c} = 1, \quad \vec{b} \cdot \vec{c} = -2$$

$$\textcircled{5} \quad 3\vec{a} \cdot (\vec{b} + 4\vec{c}) = ?$$

$$\textcircled{6} \quad (\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b}) = ?$$

$$\textcircled{7} \quad [(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}] \cdot \vec{c} = ?$$

Do these in class.

Notations:

$$\hat{i} = (1, 0, 0)$$

$$\hat{j} = (0, 1, 0)$$

$$\hat{k} = (0, 0, 1)$$

Clearly, these vectors are mutually perpendicular.

and

$$(a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$$

eg. Find the acute angle between

$$\vec{u} = \hat{i} + 2\hat{j} \quad \text{and} \quad \vec{v} = \hat{i} - \frac{1}{4}\hat{j} + 3\hat{k}$$

$$(1, 0, 0) \quad \left(\frac{1}{4}, 0, -\frac{\sqrt{3}}{4} \right)$$