Name and Student ID's: ____________________

Homework 9, Advanced Calculus 2

For problems below, we consider real normed vector spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $A \in \mathcal{L}(X, Y)$. The metrics on X and Y are norm metrics, and those on Euclidean spaces are Euclidean metrics.

1. Show that

$$\|A\| = \sup_{\|x\|_X = 1} \left\{ \|A(x)\|_Y \right\} = \sup_{x \in X, x \neq 0} \left\{ \frac{\|A(x)\|_Y}{\|x\|_X} \right\} = \inf \left\{ C \;\middle|\; \frac{\|A(x)\|_Y}{\|x\|_X} \leq C \;\forall x \in X \right\}.$$

2. Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a vector space X are called *equivalent* if there exist $A, B > 0$ so that

$$A\|x\|_1 \leq \|x\|_2 \leq B\|x\|_2 \;\; \forall x \in X.$$

   (a) Prove that two norms are equivalent if and only if they define the same topology. (i.e. a subset E is open with respect to $\|\cdot\|_1$ if and only if it is open with respect to $\|\cdot\|_2$.)

3. Prove that for every $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, there exists $y \in \mathbb{R}^n$ so that

$$A(x) = x \cdot y.$$

   Prove also that $\|A\| = \|y\|_{\mathbb{R}^n}$.