* Change of Coordinates / Parameters *

We have seen, from previous examples, that there may be different choices of local coordinates. We study how to go from one coordinate to another. ("change of coordinate", "transitions")

We first observe that choice of coordinates does not affect the smooth structure of the regular surface S.

Thm. Given a regular surface $S \subseteq \mathbb{R}^3$, $p \in S$, and

$$X_u : U \to X_u(U), \quad Y_v : V \to Y_v(V)$$

be two local coordinate around p,

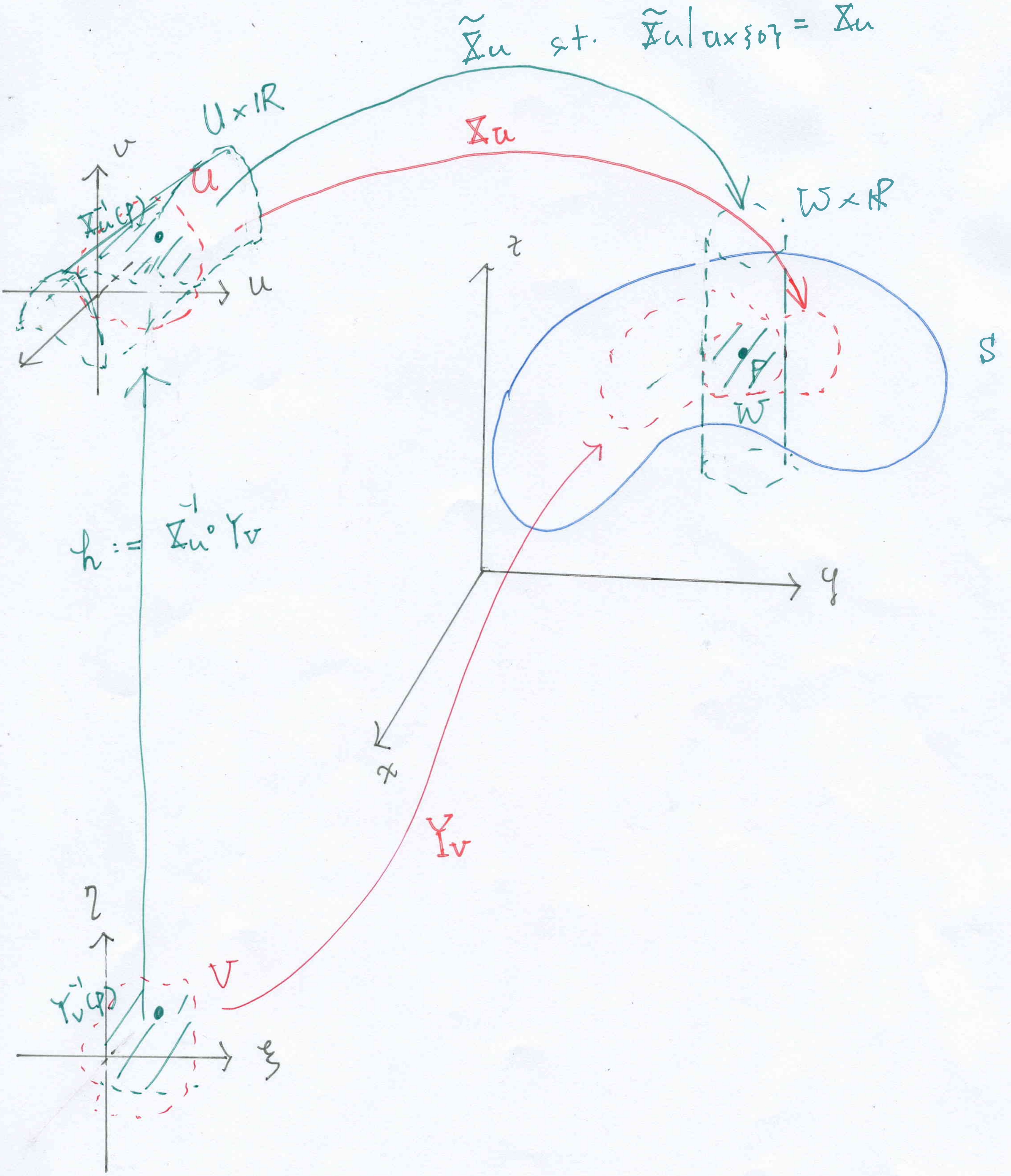
let $\underset{p\in}{W} := X_u(U) \cap Y_v$

the map $h : Y_v^{-1}(W) \to X_u^{-1}(W)$

defined by $h := X_u^{-1} \circ Y_v |_{Y^{-1}(W)}$ "transition function"

is a diffeomorphism.

-sketch-


$\tilde{\Phi}_u$ s.t. $\tilde{\Phi}_u|_{U\times\{0\}} = \Phi_u$
$U\times\mathbb{R}$
Φ_u
v
U
$\Phi_u^{-1}(p)$
u
$W\times\mathbb{R}$
z
S
p
W
$h := \Phi_u^{-1}\circ\Psi_v$
y
x
Ψ_v
η
V
$\Psi_v^{-1}(p)$
ξ

CONT

It is clear that h is bijective, and it suffices to show that h is smooth (since smoothness of $h^{-1} = Y_v^{-1} \circ X_u$ follows from identical arguments)

Note that X_u^{-1} is only defined on $X_u(U) \subseteq S$ and it's not clear what smooth functions on S are.

(let $X(u,v) = (x(u,v), y(u,v), z(u,v))$ and WLOG $\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \neq 0$)

Extend $X_u : U \to \mathbb{R}^3$ to $\tilde{X}_u : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\tilde{X}_u(u,v,t) = (x(u,v), y(u,v), z(u,v)+t)$$

and $\tilde{X}_u(u,v,0) = X_u(u,v)$ $\forall (u,v) \in U$

$$\Rightarrow \quad d\tilde{X}_u = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{pmatrix}$$

$$\Rightarrow \quad \det(\tilde{X}_u|_{(q,0)}) = \det(dX_q) \neq 0 \quad \forall q \in U$$

By inverse function thm., (and shrinking U if necessary),

$$\tilde{X}_u : U \to M := F(U) \subseteq \mathbb{R}^3$$

is diffeo. (i.e. $\tilde{X}_u^{-1}$ exists and is smooth)

and $h = X_u^{-1} \circ Y_v$

$= (\tilde{X}_u)^{-1} \circ Y_v$ $\Rightarrow$ h is smooth.

$\mathbb{R}^3 \to \mathbb{R}^2$ $\quad$ $\mathbb{R}^2 \to \mathbb{R}^3$

smooth

QED//