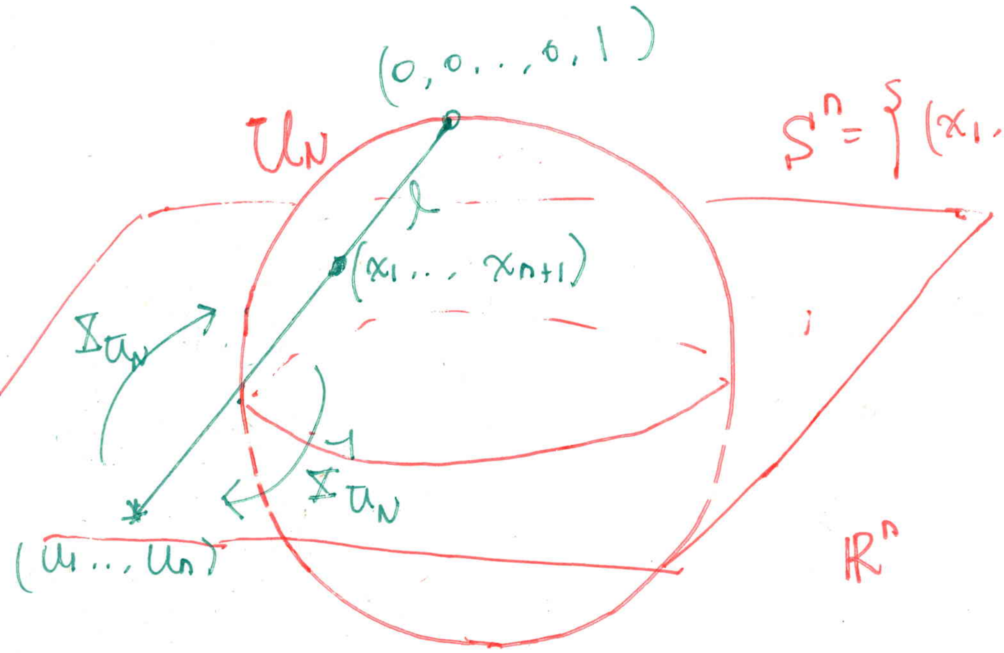
* Stereographic Projection *



$S^n = \{ (x_1, \ldots, x_{n+1}) \subseteq \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \}$

For $\underbrace{(u_1, \ldots, u_n)}_{u} \in \mathbb{R}^n$ viewed as $(u_1, \ldots, u_n, 0) \in \mathbb{R}^{n+1}$

$l : \{ (0, \ldots, 0, 1) + t(u_1 \ldots, u_n, -1) \mid t \in \mathbb{R} \}$

$l \cap S^n \Leftrightarrow t^2 \underbrace{\sum_{i=1}^{n} u_i^2}_{|u|^2} + (1-t)^2 = 1$

$\Leftrightarrow t = 0$ or $t = \dfrac{2}{|u|^2 + 1}$

$\therefore$

$\mathfrak{X}_{U_N}(u_1 \ldots u_n) = (x_1 \ldots, x_n, x_{n+1})$, where

$$x_i = \frac{2}{|u|^2 + 1} (u_1 \ldots, u_n) \; ; \; 1 \leq i \leq n$$

$$x_{n+1} = \frac{|u|^2 - 1}{|u|^2 + 1}$$

Let $(x_1, \dots x_{n+1}) \in S^n$,

$$l = \{ (0, \dots, 0, \underset{-1}{1}) + t(x_1, \dots, x_n, \underset{x_{n+1}+1}{x_{n+1}-1}) \mid t \in \mathbb{R} \}$$

$$l \cap \mathbb{R}^n \iff 1 + t(x_{n+1} - 1) = 0$$

$$\iff t = \frac{1}{1 - x_{n+1}}$$

$$\therefore \Sigma_{U_N}^{-1}(x_1 \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$$

Similarly, for $U_S = S^n \setminus \{(0, \dots, 0, -1)\}$,

connecting points with $(0, 0, \dots, 0, -1)$ gives

$$\Sigma_{U_S}(\underbrace{v_1, \dots, v_n}_{v}) = (x_1, \dots, x_n, x_{n+1})$$

with

$$x_i = \frac{2v_i}{|v|^2 + 1}, \quad 1 \le i \le n$$

$$x_{n+1} = \frac{1 - |v|^2}{|v|^2 + 1}.$$

$$\Sigma_{U_S}^{-1}(x_1 \dots x_{n+1}) = \frac{1}{1 + x_{n+1}} (x_1 \dots x_n)$$

The transition function

$$h : \Sigma_{U_N}^{-1}(U_N) \to \Sigma_{U_S}^{-1}(U_S) \quad \text{is}$$

$$h(\underbrace{u_1, \ldots, u_n}_{u}) = \Sigma_{U_S}^{-1} \cdot \Sigma_{U_N}(u_1, \ldots, u_n).$$

$$= \Sigma_{U_S}^{-1}\left(\frac{2u_1}{|u|^2+1}, \ldots, \frac{2u_n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1} \right)$$

$$= \frac{\left(\frac{2u_1}{|u|^2+1}, \ldots, \frac{2u_n}{|u|^2+1} \right)}{1 + \frac{|u|^2-1}{|u|^2+1}}$$

$$= \frac{u}{|u|^2} = \frac{u}{|u|} \cdot \frac{1}{|u|}$$

"inversion"