Note 2.2 - Properties and Computations of Limits

1 Introduction

Let's start working mathematically on limits of functions.

2 Basic Properties

A nice property of limit is that it is *linear*:

In fact, one important spirit of calculus (differentiation + integration) is to turn nonlinear things into linear ones. Even better, limit commute with all elementary algebraic operations:

We take for granted, the nice things, that for all the elementary functions introduced in Note 1,

$$\lim_{x \to c} f(x) = f(c)$$

if c is in the domain (i.e. f is defined at c). This can be a problem if we working on function of the form $\frac{p(x)}{q(x)}$ because the $q(c)$ can be 0. Here are the possibilities:

So the only interesting limits for $\frac{p(x)}{q(x)}$ at c are when *both* the top and bottom approach 0 as $x \to c$. We compute them case-by-case using elementary algebras:

For the case $c = \infty$, the limit only depends on the terms that dominate:

3 The Squeeze Theorem

A very important theorem on limit is the squeeze theorem. This is quite intuitive: if the big function and small function both approach the same limit the function in between must approach that same limit:

Here are some simple consequences:

A more important application is the following limit identity, which is used to compute the limits involving trigonometric functions:

Theorem 3.1.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Let's discuss some examples following this formula:

4 Continuity

We conclude this chapter by the concept of continuity. In plain words, it says that functions can not jump. In another words, we can always keep $f(x)$ as close as we want, as long as we keep x close:

Let's re-visit the examples we have discussed at the beginning:

All elementary functions mentioned in note 1 are continuous as long as we do not have 0 denominator. The coming chapter, the differentiation, will give us ways to distinguish these continuous functions: