Note 6.3 - Partial Fractions

1 Introduction

In this note, we deal with integrations of rational functions. These are functions that almost never have obvious antiderivatives except those like $\frac{1}{a+bx}$:

or $\frac{1}{1+x^2}$:

or some of their very simple variations. Our goal is to turn a general rational function $\frac{p(x)}{q(x)}$ into a combination of these as much as we can.

2 Algebraic Background

Let us review (or preview) some algebraic facts.

Definition 2.1. A real polynomial $p(x)$ is called *reducible* if $p(x) = q(x)r(x)$ where $deg(q) < deg(p)$. It is called *irreducible* if it is not reducible.

Turning p into qr above is called a *factorization* of p. Polynomials, like integers, can be uniquely (almost) factorized:

Theorem 2.2 (Unique Factorization)**.** *For every real polynomial $p(x)$, there exist irreducible polynomials $q_1(x), \ldots, q_n(x)$ with $deg(q_i) \leq deg(p)$ and integers $a_1, \ldots, a_n$ so that*

$$p(x) = q_1(x)^{a_1} \cdots q_n(x)^{a_n}.$$

The above factorization is unique up to a constant multiple.

Now we turn to rational function $\frac{p(x)}{q(x)}$. We may assume that $deg(p) < deg(q)$ (otherwise apply division algorithm). A theorem in algebra says that it has *partial fraction decomposition*:

Theorem 2.3. *For $\frac{p(x)}{q(x)}$ above, and $q(x) = q_1^{a_1}(x) \cdots q_n^{a_n}(x)$ be the factorization of q, the rational function can be written into*

$$\frac{p(x)}{q(x)} = \frac{r_1^1(x)}{q_1(x)} + \frac{r_2^1(x)}{q_1^2(x)} + \ldots + \frac{r_{a_1}^1}{q_1^{a_1}(x)} + \ldots + \frac{r_1^n(x)}{q_n(x)} + \frac{r_2^n(x)}{q_n^2(x)} + \ldots + \frac{r_{a_n}^n}{q_n^{a_n}(x)},$$

where $deg(r_i) < deg(q_i)$ for each i.

Of course, this theorem is only practical for low degree polynomials since we have to factor it. Let's study some example.

3 Instruction Manual

The instructions are obvious. Rewrite $\frac{p(x)}{q(x)}$ into its partial fraction decomposition and see if we can integrate that.

4 Examples