Note 6.4 - Improper Integrals

1 Introduction

We recall that ordinary integrals are defined for functions on closed intervals that are bounded. There are, however, situations when these two conditions are not satisfied. Yet, it still makes some sense to talk about integrals. We will describe them precisely and extend the definitions of integrals for them, whenever they are defined.

2 Unbounded Function

Sometimes, the functions might blow up near certain points:

However, we observe that the area underneath is getting "thin and tall" at the same time. Therefore, the area has a chance to be bounded if it gets thin faster than getting tall. To be more precise, we ask if the following limits

exist. And if they do, we define

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3 Unbounded Domain

On the other hand, we may consider functions defined on unbounded intervals:

Again, there are two factors competing each other. The area underneath is now getting "fat and short". To have a finite area, we compute

If the limit exist we define

The two integrals defined above, if existed, are called *improper integrals*.

4 The p-Integral

Let's consider a special function

$$f(x) = \frac{1}{x^p}$$

The improper integrals for these functions with various p are the basic model functions to determine convergence behaviors of many other improper integrals

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or infinite series. For $p \leq 0$ it is just a monomial which is always bounded on finite interval and will never have finite integral over unbounded domain:

So we focus on $p > 0$. For these p values, both kinds of improper integrals are interesting to discuss:

And the convergence conditions for p on these two cases are complementary to each other:

5 Comparison Theorem

As we have seen, there are many functions that are very difficult to integrate, not to mention their improper integrals. However, if we are only concerned with the convergent behaviors, monotonicity of integration take care of many situations.

Theorem 5.1. *Given* $0 \leq g(x) \leq f(x)$, *then*

Note that the sign conditions in the theorem are important:

Let's apply the theorem to discuss the finiteness of some improper integrals: