Note 1 - Introduction to Calculus and Reviews

1 Introduction

The word "calculus" has its Latin root meaning "small stones used for counting". From this point of view, we may say that calculus is a study to "count very small things". It was officially invented by Newton and Leibiniz, but many of its essential ideas can be traced back as far as ancient Greek (e.g. Archimedes's "proof" for area of a disc). Nowadays, calculus is the basic knowledge for any subject that requires any kind of quantitative reasoning skills. It is not necessary to explain why we need to study it.

The subject basically divides into

- *Differential Calculus* - from "Big" to "Small"

- *Integral Calculus* - from "Small" to "Big"

They are in fact one concept into opposite direction, as we will learn soon.

Of course, to do all these operations systematically, these quantities must be clearly given by some rules called functions.

2 Functions

Every subject in mathematics is a game of set (the "things") and logics (how things are played). Here are some common set and logic notations/examples:

A function is a rule to go from a set A to another set B. (It is actually a set, too.)

Our intuition usually associates a function with a curve called its *graph*:

Of course, not all ”curves” are graphs.

There are some ordinary operations between functions:

Here are some common elementary functions we use:

3 Inverse Functions

Some special functions $f : A \to B$ can be used to compare the "sizes" of two sets A and B. When they exist, we arrive at three types of situations:

- "$A \leq B$"

- "$A \geq B$"

- "$A \approx B$"

When $f : A \to B$ is bijective, we have a well-defined notion of *inverse function* $f^{-1} : B \to A$:

4 Exponential and Logarithmic Functions

Our earliest encounter with exponential function was something like

$$a^n = \underbrace{a \cdot \dots \cdot a}_{n \text{ times}}.$$

This only make sense for nonnegative integer n. Along the way we have perhaps seen some generalizations:

However, they still do not define numbers like $2^{\sqrt{2}}$. For 2^x with $x \in \mathbb{R}$, we might define it by

But it is not very constructive.

Let's define the exponential functions by some real life example - the compound interest.

Now that we have defined the number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, we now define

Definition 4.1 (Natural Exponential Function)**.** For $x \in \mathbb{R}$,

$$e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

You might want to check that $e^x = \underbrace{e\cdot\dots\cdot e}_{x \text{ times}}$ if $x \in \mathbb{N}$ so this is really a generalization of the baby version of exponential functions. Basic exponential laws we knew from middle school still hold:

More importantly, the function $f(x) = e^x$ from $\mathbb{R}$ to $\mathbb{R}^+$ is *bijective*:

The inverse function to e^x, from $\mathbb{R}^+$ to $\mathbb{R}$ is called the *natural logarithm* denoted by $\ln x$.

With these we can give precise definitions for numbers like $2^{\sqrt{2}}$, and the somehow mysterious log functions $\log_a b$ we have seen before:

Note that these definitions make sense even for *complex numbers*. They also give precise definitions for trigonometric functions, which we have only defined them with pictures.

Of course, these functions are not just interesting generalizations. They are one of the most important classes of functions in calculus or real world applications. They are the model functions for quantities whose growth/decay rates depend on the quantities themselves (e.g. bank account, population, bacteria, epidemics, radioactive material, signal processing ...). We will spend significant amount of time to study these functions.