Note 3.1 - Introduction to Differentiations

1 Introduction

Let's start the first main topic of calculus, the differentiation. We will define it from three different but equivalent points of view.

2 Physical Definition

In physics, we have learned the concept of velocity. It tells us how the position (y) of a particle varies after a unit of time (x):

However, it almost never tells us how particle moves at all times because experiment is always done in some finite time interval and we can only measure the *average velocity.*

But, it is generally true that smaller h at certain time x gives us more accurate description of the motion near x. To make it precise, we compute the limit

This is called the *instantaneous velocity* at x. Velocity is one kind of quantities called *rate of change*, or the change of certain dependent variable per unit change of independent variable.

3 Geometric Definition

Now let's interpret the quantities above on the graph $y = f(x)$:

We see that average rate of change between x and $x+h$ is equal to the slope of *secant line* going through $(x, f(x))$ and $(x+h, f(x+h))$. As h gets smaller, (and if everything goes well), these lines "approach" a line going through $(x, f(x))$ whose slope is *exactly* the rate of change of f at x.

4 Analytical Definition

Combining the discussion above, we define

Definition 4.1 (Differentiability and Derivative)**.**

It is not surprising that every differentiable function has to be continuous:

But continuity is not enough for differentiability:

Visually, $f(x)$ is differentiable at x if the graph is not "sharp" at $(x, f(x))$.

5 Computations From Definitions

6 Higher Order Derivatives and Leibniz Notations

The function $f'(x)$ may be differentiable again, and we denote the this derivative by $f''(x)$, which can be differentiable, too so we get $f'''(x)$, and so on. We denote by $f^{(k)}(x)$ to be the k^{th} derivative of f (with $f^{(0)} = f$).

Definition 6.1. Given $f : A \to B$, we say that $f : A \in C^k(A)$ if $f^j(x)$ exists for $j = 0, \dots, k$ for all $x \in A$. f is smooth, or $f \in C^\infty(A)$ if $f \in C^k(A)$ for all k.

Once again, all the elementary functions are smooth as long as we do not have 0 denominator. Here is an example of f that only has derivatives up to finite order.

As mentioned at the beginning, calculus was also independently invented by Leibniz. The notation of derivative $f'(x)$ he used is $\frac{df}{dx}$. This notation emphasizes more on the slope of tangent line, or the physical rate of change the function represents:

Whereas Newton's notation of $f^{(k)}$ emphasizes more on treating f as a single object on which various differential operators act on.