Note 4.2 - Mean Value Theorem and Extreme Values

1 Introduction

We discuss two central topics in the theory of differentiation. The first is extreme values including maximum and minimum value of the function in the domain. Two of the direct applications in this regard are curve sketching and optimization problems - both with rich theoretical and practical values.

The second topic is the mean value theorem, a result that generalizes our common sense on the concept of averages. Mean value theorem is often used to prove other important and interesting result about the behaviors of a function (such as monotonicity, existence of roots... etc).

2 Monotonic Functions

Monotonic functions are those functions that keep going up or going down. They are classified into the following types:

For differential functions, monotonicity is characterized by tangent lines:

These are determined precisely by the signs of the first derivatives:

It is obvious that points where first derivatives are zero deserve some attention.

3 Extreme Values

The extreme values of a function over a domain are always interesting to discuss. Let's define them first.

Definition 3.1. Given $f : A \to B$ and $c \in A$, we say that
f has *maximum* at c if

f has *minimum* at c if

Extreme values do not always exist. The nonexistence is either due to the defect of function f:

or the domain A:

Here is an important theorem that guarantees the existence of extreme values:

Theorem 3.2 (Extreme Value Theorem "EVT"). *A* continuous *function on a* closed *interval* $[a, b]$ *always has maximum and minimum.*

Despite the EVT, it is not always easy to locate and find the extreme values. There is, however, a weaker version of extreme values:

Definition 3.3. Given $f : A \to B$ and $c \in A$, we say that

f has *local maximum* at c if

f has *local minimum* at c if

Clearly, extreme values are local extreme values. We sometimes refer to them as the *global* (or *absolute*) maximum or minimum. The general strategy is therefore to look for absolute extreme values among local ones.

4 Classification of Local Extreme Values

Now we discuss strategies to capture local extreme values. Let's assume that f is at least continuous.

We see that local extremes occur at points where monotonicities change. To be more precise, we have

These can be observed more easily if f is differentiable. Since monotonicity is equivalent to the sign of f', local extremes occur at points where f' change signs (i.e. f'=0). It is important to note that having $f'(x_0) = 0$ does NOT mean f has local max/min at x_0:

However, a local max/min for a differentiable function must occur at points with zero derivative. These points are therefore *suspects* for local extreme values and we look for the *guilty* ones among them.

For functions that are not differentiable everywhere, we have no information of derivative at "bad points" and we make them all suspects. These points (points with zero or no derivative) are called *critical point* of f. We look for local extreme values from them.

Let's discuss some examples.

5 The Mean Value Theorem

We all have some common sense of "mean", for example the average score of exam scores of a class. Either everybody scores the same (then this score is the mean), else there will be at least two students scoring above and below the average. In another words, the average can not be higher (or lower) than everyone's score.

Now if we have a continuum of student and the exam score is a continuous distribution of real numbers, then it means that someone must scores *exactly* the mean. The mean value theorem is basically this idea applied to the quantity $f'(s)$. (It is actually stronger than what we describe above.)

Theorem 5.1 (Mean Value Theorem MVT)**.** *Given $f : [a,b] \to \mathbb{R}$ differentiable on (a,b), there is a point $c \in (a,b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

One can interpret physically by saying that a particle moving on some time period must travel, at some time, exactly as fast as the average velocity.

The mean value theorem can be deduced from a special case called the *Rolle's Theorem*:

Theorem 5.2 (Rolle's Theorem)**.** *Given $f : [a,b] \to \mathbb{R}$, differentiable on (a,b), so that $f(a) = f(b) = 0$, there is a point $c \in (a,b)$ so that $f'(c) = 0$.*

Proof. □

Rolle's Theorem then implies the mean value theorem:

Mean value theorem and Rolle's theorem prove a number of algebraic facts: