Note 5.2 - The Fundamental Theorem of Calculus

1 Introduction

The motivation to study the fundamental theorem is probably to compute integration. But the theorem itself is probably the most important and fundamental fact to know in calculus.

2 The Area Functions

Given an continuous function $f(x)$ and some $a \in \mathbb{R}$, we know that $\int_a^x f(x)\, dx$ is always defined for $x \geq a$. For $x < a$, we define

$$\int_a^x f(x)\, dx := -\int_x^a f(x)\, dx.$$

This gives us a well-defined *area function*

$$F(x) = \int_a^x f(x)\, dx$$

on $\mathbb{R}$. We have the following properties that follows directly from definition of F:

If we can explicitly find this function F, the tedious computations in Note 5.1 become much simpler:

So how to find this F?

3 The Fundamental Theorem of Calculus

We now come to the most important theorem of the entire course: finding the relationship between F and f described above.

Theorem 3.1 (Fundamental Theorem of Calculus FTC)**.** *Given a continuous function f and*

$$F(x) := \int_s^x f(t)\, dt$$

for some $s \in \mathbb{R}$, then

- *F is differentiable and $F' = f$*
- $$\int_a^b f(x)\, dx = F(b) - F(a).$$

So this gives us much more clues on how to find F. We will derive (more precisely, guess) some formula in the next note.

Here is a more general form of FTC:

Theorem 3.2 (Generalized Fundamental Theorem of Calculus)**.** *Given a continuous function f and differentiable functions g, h, let*

$$F(x) = \int_{h(x)}^{g(x)} f(t)\, dt.$$

Then,

$$F'(x) = f(h(x))h'(x) - f(g(x))g'(x).$$

It is an easy consequence of the chain rule:

4 Examples