Note 8.1 - Introduction to Sequences

1 Introduction

In the three following notes, our goal is to approximate any smooth function into a function involving only addition and multiplication (i.e. polynomials). The approximation will be performed in ways so that the errors approach 0 as the degrees of polynomials approach ∞. In appropriate cases, these approximations work well with differentiations and integrations.

To conclude, we are turning smooth and continuous things into discrete things (ones we can count). Let us begin by studying countable things called *sequences*.

2 Definitions and Examples

Definition 2.1. A *sequence* of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$.

3 Convergence

We are often most interested in whether the list of number $\{a_n\}$ *approach* something as $n \to \infty$. This is defined precisely by

Definition 3.1.

4 Properties

The algebraic properties of lim hold for sequences:

For real sequence, there is very important characterization of convergent sequence. Let's first define

Definition 4.1. A sequence $\{a_n\}$ is *bounded* if there is M so that $|a_n| \leq M$ for all n.

Definition 4.2. A sequence is called monotonic if it is nondecreasing ($a_n \leq a_{n+1}$ for all n) or nonincreasing ($a_n \geq a_{n+1}$ for all n).

The theorem is

Theorem 4.3. *A bounded, monotonic sequence of real numbers is convergent.*

5 Calculus on Sequences

Clearly, we can not perform calculus on sequences as they are function defined on $\mathbb{N}$ with elements that are at least 1 unit apart. Nevertheless, If there exist a continuous or smooth function on $\mathbb{R}$ that covers $\{a_n\}$:

then many properties of f are true for $\{a_n\}$:

6 Examples