Note 8.2 - Introduction to Series

1 Introduction

A series is a sequence defined from a sequence. It is the discrete version of integration that bears many properties in common.

2 Definition and Examples

Definition 2.1. Given a sequence $\{a_k\}$, we define a series $\{s_n\}$ by

$$s_n = \sum_{k=k_0}^{n} a_k.$$

Each s_n is called the n^{th} *partial sum*. We say that the series converges if the partial sums converge and denote the limit by

It is usually not easy to determine whether a series converges. However, there are instances that series obviously diverge:

Theorem 2.2 (The k^{th} Term Test)**.** *In the notations above, if the series diverges, then* $a_k \to 0$

In the other words, if $a_k \nrightarrow 0$, then the series diverges.

Even if the series converges, very often we have no idea what value it converges to.

3 Elementary Examples

We are probably familiar with this series. Given $a, r \in \mathbb{R}$, let $a_k = ar^k$. We have $s_0 = a$ and

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

for $n \geq 1$.

It is then not hard to tell that s_n converges if and only if $|r| < 1$ and

$$s = \lim_{n \to \infty} s_n = \frac{a}{1 - r},$$

The other series with easy computable limit is called the *telescoping series*:

4 Convergence Tests

As mentioned before, we are often more concerned on whether the series converge over the actual limiting value. Some of them are easier to apply than the others, but often come with the tradeoff of applicability or conclusiveness. Let's skim through them below.

5 Power Series

Power series is a polynomial of infinite degree. It is formally written as

$$P(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Evidently, the convergence behavior of P depends on the value of x. It certainly converges for $x = 0$, but can diverge for other x's:

It is a theorem that a power series always converge on an interval $(-r, r)$, called *convergence interval*, for $r \in (0, \infty]$. This interval is usually determined by ratio or root test:

On the next and final note, we will study a power series of particular importance.