Note 9.2 - Elementary Differential Equations

1 Introduction

Differential equations is a vast subject in mathematics whose aspects stretch from theories to applications. It studies functions involving functions and their derivatives. One of the main goals is to *solve* for the functions from many different approaches. Unfortunately, explicit solutions are usually very difficult (if not impossible) to find, even knowing the existence (and uniqueness) of them. Various numerical and approximation methods are developed when one can not explicitly solve the equations. In this note, we look at a few very basic problems that can be solved explicitly, as well as the numerical method for approximating solutions.

2 Separable Equations

We have seen that exponential function solves the equation $y' = ky$. Here, we generalize this equation to equation of the form

$$\frac{dy}{dx} = f(x)g(y).$$

The way to solve it is quite obvious: we just integrate back.

Let's study some examples.

3 Logistic Equations

A very important example of separable equations is the logistic equation. This is a more realistic equation to describe a population in some environment than a pure exponential function $y = y_0 e^{kt}$. In real world, an environment usually can support only a finite amount of a population due to resources, conflicts, or other restrictions. We call this limit the *carrying capacity.* With this in mind, we consider the following differential equation for a population P

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where r is the unconstrained growth rate and K is the carrying capacity. Here are some quick observations:

Let try to solve for P:

4 Examples and More Discussions

5 First Order Linear Differential Equations

We study the differential equations of the form

$$\frac{dy}{dx} + a(x)y = b(x)$$

for some functions $a(x)$,$b(x)$. It is, of course, not linear in rigorous sense, meaning that y_1, y_2 are solutions then ay_1 and $y_1 + y_2$ are both solutions (unless $b(x) = 0$). A typical way to solve this equation is by the method of *integrating factor*:

Let's study some typical electric circuits that are basic models for many appliances we use nowadays.

6 Euler's Methods

Euler's methods is a classical way to find approximated solution of a first order differential equation of the form

$$y' = F(x, y)$$

using the very basic idea of "connecting data points with straight lines". Since y' above represents the slope of the solution at various points, we can generate a *slope fields* on $xy-$plane, and our solutions are tangent to these short lines:

Systematically, here is how we generate approximated solutions:

Euler's methods, although very basic and intuitive, is in fact the central idea for proving the existence and uniqueness of solutions to differential equations.