Note 2.2 - Properties and Computations of Limits

1 Introduction

Let's start working mathematically on limits of functions.

2 Basic Properties

A nice property of limit is that it is *linear*:

In fact, one important spirit of calculus (differentiation + integration) is to turn nonlinear things into linear ones. Even better, limit commute with all elementary algebraic operations:

We take for granted, the nice things, that for all the elementary functions introduced in Note 1,

$$\lim_{x \to c} f(x) = f(c)$$

if c is in the domain (i.e. f is defined at c). This can be a problem if we working on function of the form $\frac{p(x)}{q(x)}$ because the $q(c)$ can be 0. Here are the possibilities:

So the only interesting limits for $\frac{p(x)}{q(x)}$ at c are when *both* the top and bottom approach 0 as $x \to c$. We compute them case-by-case using elementary algebras:

For the case $c = \infty$, the limit only depends on the terms that dominate:

3 The Squeeze Theorem

A very important theorem on limit is the squeeze theorem. This is quite intuitive: if the big function and small function both approach the same limit the function in between must approach that same limit:

Here are some simple consequences:

A more important application is the following limit identity, which is used to compute the limits involving trigonometric functions:

Theorem 3.1.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Let's discuss some examples following this formula:

4 Continuity

We conclude this chapter by the concept of continuity. In plain words, it says that functions can not jump. In another words, we can always keep $f(x)$ as close as we want, as long as we keep x close:

Let's re-visit the examples we have discussed at the beginning:

All elementary functions mentioned in note 1 are continuous as long as we do not have 0 denominator. The coming chapter, the differentiation, will give us ways to distinguish these continuous functions:

5 Existence of Limit

We have seen that the function $f(x) = \frac{x}{|x|}$ does not have a limit at 0. However, there are weaker versions of limits if we only require *part* of the x's near c to satisfy the limit definitions.

Clearly, limit exists if and only if these weaker limits exist and are equal.

6 General Notions of Limit and Continuity

We briefly discuss the generalizations of limit and continuity for functions of several variables $f : \mathbb{R}^n \to \mathbb{R}$. Let's re-examine the intuitive definition for limit of single variable functions and determine what to generalize.

The property "$f(x)$ closes to L" needs no revision, but "x closes c" needs to be generalized.

With these, we have the following intuitive and precise definitions of limits.

The existence of limit, and continuity, are much more complicated to prove for multi-variable functions since there are *infinitely many* ways for $\mathbf{x}$ to approach $\mathbf{c}$. However, it is often easier to prove the *nonexistence* of limit (and therefore the discontinuity).

Even more abstractly, we note that all these limit definitions consist of relations between various *distances*. In mathematical analysis, we call a set with the structure of distances *metric spaces*, where "distances" are referred to as "metrics". Limits can therefore be naturally defined for functions between metric spaces: