Note 3.5: Partial Derivatives

1 Introduction

For functions between $\mathbb{R}^n$ and $\mathbb{R}^m$, rate of change becomes much more subtle. It is not clear *which* quantity we are measuring the change. Moreover, every variable may contributes changes of functions and therefore the term *rate* is not well defined. There is a general notion of derivative for these functions. However, it requires knowledge of linear algebra and is not suitable to discuss here. We discuss the special case with $m = 1$.

2 Definitions

From previous illustrations, we now define the *partial derivatives* of a functions from $U \subset \mathbb{R}^n$ to $\mathbb{R}$.

3 Computations

From its definitions, it is clear that $\frac{\partial f}{\partial x_i}$ is computed simply by treating all other x_j's as constant:

Sometimes, partial derivatives have to computed from definition, if functions are not defined by one simple formula:

An important fact that is not observed for single-variable functions is that existence of partial derivatives *does not* imply continuities:

This observation implies that existence of partial derivatives, without other conditions, generally doesn't mean much. As such, the definition "differentiable" is more sophisticated (and requires, again, linear algebra) than existence of all partial derivatives.

4 Higher Order Derivatives

Just like single variable functions, it is possible to consecutively differentiate a function. Since there are more than one choice of variable to differentiate against, we have more possible higher derivatives.

An important question to ask is that whether the *order* of variables to take derivatives is important or not. It turns out taking derivatives with respect to the same set of variables in different order *may* produce different result:

However, with certain additional conditions, we may take higher derivatives in any order: