Note 4.5 - Other Applications of Differentiations

1 Introduction

We wrap up the topic of differentiation with two additional applications: The l'Hopital's rule and the introduction to Newton's method.

2 Indeterminate Forms and l'Hopital's Rule

We have seen limits of the form $\frac{0}{0}$, where anything can happen. It is a competition between denominator and numerator on each one's rate of approaching 0. Similar situations occur in limits of the forms $0 \cdot \infty$, $\frac{\infty}{\infty}$, 1^∞, or ∞^0. The existences of these limits (and their values if existed), are heavily dependent on the rates for each component to approach their respective limit, and there are highly related to their derivatives (if differentiable). This is the principle of l'Hopital's rule.

Theorem 2.1 (l'Hopital's Rule)**.** *Given functions* f, g *differentiable at* $x = c$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$*, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Note that c here may be ∞ and the rule still holds.

Let's practice some basic examples

We explain, rather non-rigorously, that l'Hopital's rule applies to the case

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty.$$

Some examples:

Here are examples of other indeterminate forms. We deal with them by turning them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Note that one may *not* use l'Hopital's rule if the limits are not of indeterminate forms:

3 The Newton's Method

Newton's method is a numerical way to approximate the root of a function by iterative intersection of tangent lines with x-axis:

Let's use this method to approximate $\sqrt{2}$.

The method seems to be better than the method of bisection. However, it comes with the drawback that the method often fails. Either the function does not have the right derivative:

More serious is that the success of the approximation depends on the initial guess, and there is no easy way to tell if the guess is a good one: