Note 5.1 - Introduction to Integrations

1 Introduction

We come to the second main topic of calculus: the integration. Unlike differentiations, integration is motivated from very basic questions of approximating Arclength and area of irregular shapes.

2 Approximation of Area

Here is a problem we probably have all seen before: find the area of the disc.

What we do in elementary school is to place the disc on a grid paper like following and count how many square it occupies:

Of course, there are grids that only partially filled. We can either *ignore* them or count them as one. These give two approximations, one below the actual are and one above:

To improve the estimate, we divide the paper into more grids (this is called *refinement* and repeat the estimate.

We still get two estimates with the actual area in between. But notice that we are ignoring less area and add less extra area into the estimates, making the two bounds closer.

So we keep refining the grids, making the lower bound greater and upper bound smaller. Hopefully, in the limit, they approach the same value. Since the actual area is in between them throughout the processes, that value must be the actual area.

3 Precise Mathematical Formulation

We now discuss this process with precise mathematical languages. Consider

$$f : [a, b] \to \mathbb{R}$$

that is, say, continuous (for now).

We want to compute the area under $y = f(x)$ over $[a, b]$. We follow the same principle by covering the area with *rectangles* that are always smaller or larger than the actual area (but with their gaps getting smaller).

For each partition P of $[a, b]$, we get lower bound $L(P, f)$ and upper bound $U(P, f)$ with the properties

So we keep refining P, therefore keep increasing $L(P, f)$ and keep decreasing $U(P, f)$, and hopefully, their "limit" agree eventually.

Note that we do not need functions to be continuous to satisfy integrability.

There is also an important theorem, saying that the "limit" mentioned above does not have to be a sequence of refinements. It suffices to take a sequence of partitions $\{P_n\}$ so that the maximum length of the subdivisions (the *mesh* approach 0.

4 Some Basic Computations

Let's do some examples, starting with ones we already know the answers:

Or some we can not compute using elementary math

We see that even for the a simple function like $f(x) = x^2$, the integral can be quite tricky to compute. We clearly need better formula to compute integration, at least for continuous functions.