Note 7.4 - Physical Applications

1 Introduction

In this note, we use the language of calculus to describe the physical concepts of mass/density, center of mass and moment of inertia, and work that are not necessarily uniform.

2 Mass and Density

Mass is usually understood to be a quantity of how *heavy* an object is. More precisely, it measures how difficult an object *accelerates* when a force is applied on it. Another intuitive concept of an object is how much *space* it occupies. They are usually known as length, area, or volume in dimension 1, 2, and 3, respectively. Associated with these, we can measure how *dense* an object. This is a quantity that describes how much mass per unit space:

$$\textit{Density} = \frac{\textit{Mass}}{\textit{Length, Area, or Volume}}.$$

Of course, masses are usually not uniformly distributed throughout the object and the quantity above in general does not accurately describe how dense an object is at a given point. Therefore, we *differentiate* mass with respect to the spaces. Let's focus on dimension 1 and define *line density function* $\rho(x)$:

3 Moments and Center of Mass

Many physical laws of motion, such as $F = ma$, describe system of *point masses* occupying no spaces. However, real world objects are of finite sizes and motions *within* themselves (e.g. rotations, spin) occur. These are measured by something called *moments*. Center of mass is a point where moments cancel out, and therefore there is no *angular acceleration* at that point. In another words, when a force is applied on the object, the motion of its center of mass stays the same as if we apply all the force and concentrate all the mass on this point. Let's start with a system of finitely many point masses:

We then apply the concepts to a finite object with continuous density distribution:

As we have seen, for objects with certain symmetries, centers of masses are usually located at axes of symmetries. A particular kind of symmetric object are those formed by rotations. For these objects, their volumes can be computed quite conveniently:

Theorem 3.1 (Theorem of Pappus)**.**

4 Work

We are familiar with the concept of *energy*, a classically conserved quantity for a physical system to perform various physical procedures (e.g. motions, heating, water flows, light emission, ... etc). Energy is transferred from a system to another in the form of *work*. For classical motions, work is defined by

$$W = \mathbf{F} \cdot \mathbf{S},$$

where $\mathbf{F}$ is the force applied and $\mathbf{S}$ is the displacement. In dimension one, the product above is just the ordinary multiplication. Again, we have the usual problem that the force varies as the object moves along:

5 Examples