Note 8.1 - Introduction to Sequences

1 Introduction

In the three following notes, our goal is to approximate any smooth function into a function involving only addition and multiplication (i.e. polynomials). The approximation will be performed in ways so that the errors approach 0 as the degrees of polynomials approach ∞. In appropriate cases, these approximations work well with differentiations and integrations.

To conclude, we are turning smooth and continuous things into discrete things (ones we can count). Let us begin by studying countable things called *sequences*.

2 Definitions and Examples

Definition 2.1. A *sequence* of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$.

3 Convergence

We are often most interested in whether the list of number $\{a_n\}$ *approach* something as $n \to \infty$. This is defined precisely by

Definition 3.1.

4 Properties

The algebraic properties of lim hold for sequences:

For real sequence, there is very important characterization of convergent sequence. Let's first define

Definition 4.1. A sequence $\{a_n\}$ is *bounded* if there is M so that $|a_n| \leq M$ for all n.

Definition 4.2. A sequence is called monotonic if it is nondecreasing ($a_n \leq a_{n+1}$ for all n) or nonincreasing ($a_n \geq a_{n+1}$ for all n).

The theorem is

Theorem 4.3. *A bounded, monotonic sequence of real numbers is convergent.*

5 Series

Definition 5.1. Given a sequence $\{a_k\}$, we define a series $\{s_n\}$ by

$$s_n = \sum_{k=k_0}^{n} a_k.$$

Each s_n is called the n^{th} *partial sum*. We say that the series converges if the partial sums converge and denote the limit by

It is usually not easy to determine whether a series converges. However, there are instances that series obviously diverge:

Theorem 5.2 (The k^{th} Term Test)**.** *In the notations above, if the series diverges, then* $a_k \to 0$

In the other words, if $a_k \nrightarrow 0$, then the series diverges.

Even if the series converges, very often we have no idea what value it converges to.

6 Elementary Examples

We are probably familiar with this series. Given $a, r \in \mathbb{R}$, let $a_k = ar^k$. We have $s_0 = a$ and

$$s_n = \frac{a(1-r^n)}{1-r}$$

for $n \geq 1$.

It is then not hard to tell that s_n converges if and only if $|r| < 1$ and

$$s = \lim_{n\to\infty} s_n = \frac{a}{1-r},$$

The other series with easy computable limit is called the *telescoping series*:

7 Convergence Tests

As mentioned before, we are often more concerned on whether the series converge over the actual limiting value. Some of them are easier to apply than the others, but often come with the tradeoff of applicability or conclusiveness. Let's skim through them below.

8 Power Series

Power series is a polynomial of infinite degree. It is formally written as

$$P(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Evidently, the convergence behavior of P depends on the value of x. It certainly converges for $x = 0$, but can diverge for other x's:

It is a theorem that a power series always converge on an interval $(-r, r)$, called *convergence interval*, for $r \in (0, \infty]$. This interval is usually determined by ratio or root test:

On the next and final note, we will study a power series of particular importance.