Note 8.3 - Taylor and Maclaurin Series

1 Introduction

We come to the final note of the course with a practical question: how does a calculator "compute" values like $\sin 2.3$ or $e^{0.87}$? Physically, a calculator only knows how to add (subtract) and multiply (divide) by the designs of *logical circuit*. That is, the only *functions* calculator really knows are polynomials and all other functions are *approximated* by them.

2 Approximations by Polynomials

We have seen the approximation of a function f at c by its tangent line, which is degree 1 polynomial p_1 that agrees with f at c to the first order:

Geometrically, this p_1 pass through $(c, f(c))$ and has the same monotonicity as f near c:

We might want to improve the approximation by a polynomial that also has the same concavity as f near c. To be more precise, we want a degree 2 polynomial $p_2(x)$ so that

This polynomial is not very hard to construct:

It is then quite natural to expect a *sequence* of polynomials $\{p_k(x)\}$. For each k, p_k agrees with f at c to order k:

Let's construct these $p_k(x)$:

Let's look at the following example $f(x) = e^x \sin x$. (See slides)

From the example, we observe two general trends:

These observations are evidence of the very important

Theorem 2.1 (Taylor Theorem)**.**

3 Taylor and Maclaurin Series

Take $c = 0$, we derive the four basic Taylor series of smooth functions. They are also known as the Maclaurin series.

3.1 e^x

3.2 $\sin x$ and $\cos x$

3.3 $\ln x$

3.4 The Binomial Theorem

4 Some Simple Variations

There are a few simple manipulations we can perform on known Taylor series to generate new ones:

The more subtle ones are term-by-term differentiations and integrations. Power series does not always act so nicely with calculus like polynomials:

However, it can be proved that term-by-term calculus is valid within the radius of convergence (excluding endpoints). We may then generate more Taylor/Maclaurin series.