On the Dirac-Klein-Gordon
Equations in Three Space Dimensions

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Abstract. We establish a local existence result for Dirac-Klein-Gordon equations in three space dimensions, employing a null form estimate and a fixed point argument.

0. Introduction and Main Results.

In the present work, we like to study the Cauchy problem for the Dirac-Klein-Gordon equations. The unknown quantities are a spinor field $\psi : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$. The evolution equations for the fields are given below,

$$\mathcal{D}\psi = \phi\psi; \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^3 \tag{0.1a}$$

$$\Box\phi = \overline{\psi}\psi; \tag{0.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_{,t}(0,x) = \phi_1(x), \tag{0.1c}$$

where $\mathcal{D}$ is the Dirac operator, $\mathcal{D} := -i\gamma^\mu \partial_\mu$, $\mu = 0,1,2,3$, and γ^μ are the Dirac matrices, the wave operator $\Box = -\partial_{tt} + \Delta$, and $\overline{\psi} = \psi^\dagger \gamma^0$, and † is the complex conjugate transpose. The matrices can be written as follows. First let us define the 2×2 matrices σ_1, σ_2, and σ_3,

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{0.2a}$$

The matrices γ^μ are defined via

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}. \tag{0.2b}$$

The purpose of this work is to demonstrate a variant null form estimate, by employing the solution representations in Fourier transform of the DKG equations. Such estimate can improve the existence result, see [Bo]. We will take advantage of the null form structure depicted in the nonlinear term $\overline{\psi}\psi$, see [KM] and [Bo].

For the DKG system, there are many conserved quantities which are not positive definite, such as the energy, the momenta, and the angular momenta. However there is a known positive conserved quantity which is the law of conservation of charge, $\int |\psi(t)|^2 \, dx \; = \text{constant}$, see [GS]

In '74, Chadam and Glassey showed that the Cauchy problem for the DKG equations has a unique local solution for $\psi_0 \in H^2$, $\phi_0 \in H^2$, $\phi_1 \in H^1$, and global solution for a particular class of initial data, see [CG]. In '81, Choquet-Bruhat proved the global existence result for the (massless) DKG equations by assuming small data, see [CB]. In '88, Bachelot gave the global existence for DKG system with small data, see [Ba]. In '99, Bournaveas derived a local existence for the DKG equations, based on a null form estimate, if $\psi_0 \in H^{\frac{1}{2}}$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [Bo].

The outline of this paper is as follows. First we derive some solutions representations in Fourier transform. Next we prove some a priori estimates of solutions for Dirac equation and for wave equation. Then we show a local existence result for (0.1), employing the null form estimate together with some other estimates, and a fixed point argument. Finally we show the key estimate, namely the null form estimate.

The main result in this work is as follows.

Theorem 0.1. *(Local Existence) Let* $\epsilon > 0$*. If the initial data of (0.1)* $\psi_0 \in H^{\frac{1}{4}+\epsilon}(\mathbb{R}^3)$, $\phi_0 \in H^1(\mathbb{R}^3)$, $\phi_1 \in L^2(\mathbb{R}^3)$, *then there is a unique local solution for (0.1).*

Remarks.

1. The *DKG* equations follow from the Lagrangian

$$\int_{\mathbb{R}^{3+1}} \Big\{ |\nabla\phi|^2 - |\phi_t|^2 - \overline{\psi}\mathcal{D}\psi - \phi\overline{\psi}\psi \Big\} dxdt. \tag{0.3}$$

2. The Dirac-Klein-Gordon system must be

$$\begin{cases} \mathcal{D}\psi = \phi\psi; \\ \Box\phi + m^2\phi = \overline{\psi}\psi, \end{cases} \tag{0.4}$$

and the proof works for this system too.

3. Let I be the 4×4 identity matrix, $\widehat{\mathcal{D}} = \gamma^0\tau + \gamma^j\xi_j$, and $\widehat{\Box} = \tau^2 - |\xi|^2$, thus we have $\widehat{\mathcal{D}}^2 = \widehat{\Box} I$.

4. $\overline{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$, where ψ_j are the component functions of the vector function ψ, which take values in $\mathbb{C}$.

1. Solution Representation.

In what follows, we denote by (t, x) the time-space variables and by (τ, ξ) the dual variables with respect to the Fourier transform of a given function. We will use $\mu = \frac{1}{4} + \epsilon$, $\alpha = \frac{1}{4} + \delta$, and $\nu = \epsilon - \delta$ throughout the paper. We will also often skip the constant in the inequalities. For convenience, we denote the multipliers by

$$\widehat{E}(\tau, \xi) = |\tau| + |\xi| + 1, \tag{1.1a}$$

$$\widehat{S}(\tau, \xi) = \Big||\tau| - |\xi|\Big| + 1, \tag{1.1b}$$

$$\widehat{W}(\tau, \xi) = \tau^2 - |\xi|^2, \tag{1.1c}$$

$$\widehat{D}(\tau, \xi) = \gamma^0\tau + \gamma^j\xi_j, \quad j = 1, 2, 3, \tag{1.1d}$$

$$\widehat{M}(\xi) = |\xi| + 1. \tag{1.1e}$$

Notice that $\widehat{W}$ and $\widehat{D}$ are the symbols of the wave and Dirac operators respectively. Also there is a summation over the upper and lower indeices.

Consider the Dirac equation,

$$\begin{cases} \mathcal{D}\psi = G, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^3, \\ \psi(0) = \psi_0. \end{cases} \tag{1.2}$$

First by taking the Fourier transform on (1.2) over the space variable and solving the resulting ODE, we can formally write down the solution as follows.

$$\widetilde{\psi}(t,\xi)=\frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)\gamma^0\widehat{\psi}_0(\xi)+\frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{\psi}_0(\xi)+\\ \int_0^t\frac{e^{i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)i\widetilde{G}(s,\xi)\,ds+\int_0^t\frac{e^{-i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)i\widetilde{G}(s,\xi)\,ds. \tag{1.3}$$

Rewriting the inhomogeneous terms in (1.3) gives

$$\widetilde{\psi}(t,\xi)=\Big[\frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)+\frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\Big]\gamma^0\widehat{\psi}_0(\xi)+\\ \int\Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}(\tau,\xi)d\tau. \tag{1.4}$$

Now we split the function $\widehat{G}$ into several parts in the following manner. Consider $\widehat{a}(\tau)$ a cut-off function equals 1 if $|\tau|\le\frac{1}{2}$ and equals 0 if $|\tau|\ge 1$, and denote by $h(\tau)$ the Heaviside function. For simplicity, let us write

$$\widehat{G}_\pm(\tau,\xi):=h(\pm\tau)\widehat{a}(\tau\mp|\xi|)\widehat{G}(\tau,\xi), \tag{1.5a}$$

$$\widehat{G}_f(\tau,\xi):=\widehat{G}(\tau,\xi)-\big(\widehat{G}_+(\tau,\xi)+\widehat{G}_-(\tau,\xi)\big), \tag{1.5b}$$

$$\widehat{D}_\pm:=\widehat{D}(|\xi|,\pm\xi). \tag{1.5c}$$

Notice that $\widehat{G}_\pm$ are supported in the regions $\{(\tau,\xi):\pm\tau>0,|\tau\mp|\xi||\le 1\}$ respectively. Using the decomposition of the forcing term $\widehat{G}=\widehat{G}_f+\widehat{G}_++\widehat{G}_-$, the inhomogeneous term in (1.4) can be written as

$$\int\Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}_f(\tau,\xi)d\tau\\ =\int e^{it\tau}\frac{\widehat{D}(\tau,\xi)}{\tau^2-|\xi|^2}\widehat{G}_f\,d\tau-e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{\widehat{G}_f}{\tau-|\xi|}d\tau-\\ e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int\frac{\widehat{G}_f}{\tau+|\xi|}d\tau, \tag{1.6a}$$

$$\int\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}_+(\widehat{G}_++\widehat{G}_-)d\tau\\ =e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{e^{it(\tau-|\xi|)}-1}{\tau-|\xi|}(\widehat{G}_++\widehat{a}_6(\tau)\widehat{G}_-)d\tau+\\ \int e^{it\tau}\frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}d\tau-e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{(1-\widehat{a}_6(\tau))\widehat{G}_-}{\tau-|\xi|}d\tau, \tag{1.6b}$$

where $\widehat{a}_6(\tau) = \widehat{a}(\frac{\tau}{6})$ and $\widehat{a}$ is the cut-off function defined previously.

$$\begin{aligned}\int \frac{e^{it\tau} - e^{-it|\xi|}}{2|\xi|(\tau + |\xi|)} \widehat{D}_-(\widehat{G}_+ + \widehat{G}_-) d\tau & \\ = e^{-it|\xi|} \frac{\widehat{D}_-}{2|\xi|} \int \frac{e^{it(\tau+|\xi|)} - 1}{\tau + |\xi|} (\widehat{a}_6(\tau)\widehat{G}_+ + \widehat{G}_-) d\tau + & \\ \int e^{it\tau} \frac{(1 - \widehat{a}_6(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau + |\xi|)} d\tau - e^{-it|\xi|} \frac{\widehat{D}_-}{2|\xi|} \int \frac{(1 - \widehat{a}_6(\tau))\widehat{G}_+}{\tau + |\xi|} d\tau. & \quad (1.6c)\end{aligned}$$

Recall the power expansion

$$e^{it(\tau \pm |\xi|)} - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} (it)^k (\tau \pm |\xi|)^k. \tag{1.7}$$

Combining (1.4)-(1.7), we can give a formula for $\widehat{\psi}$, namely

$$\widehat{\psi}(\tau, \xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau, \xi) \widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau, \xi) \widehat{A}_{-,k}(\xi) \Big) + \widehat{K}(\tau, \xi), \tag{1.8}$$

where $\delta_\pm(\tau, \xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau, \xi) := \frac{\widehat{D}(\tau, \xi)}{\widehat{W}(\tau, \xi)} \widehat{G}_f + \frac{(1 - \widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau - |\xi|)} + \frac{(1 - \widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau + |\xi|)}, \tag{1.9a}$$

$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|} \Big[\gamma^0 \widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1 - \widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda \mp |\xi|} d\lambda \Big], \tag{1.9b}$$

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm (-1)^k}{2|\xi| k!} \int (\lambda \mp |\xi|)^{k-1} \big[\widehat{G}_\pm + \widehat{a}_6(\lambda) \widehat{G}_\mp \big] d\lambda. \tag{1.9c}$$

Consider the wave equation,

$$\begin{cases} \Box\phi = F, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^3, \\ \phi(0) = \phi_0, \ \ \phi_t(0) = \phi_1. \end{cases} \tag{1.10}$$

Taking Fourier transform on (1.10) and solving the resulting ODE gives

$$\widetilde{\phi}(t, \xi) = \cos t|\xi| \widehat{\phi}_0(\xi) + \frac{\sin t|\xi|}{|\xi|} \widehat{\phi}_1(\xi) - \int_0^t \frac{\sin{(t - s)|\xi|}}{|\xi|} \widetilde{F}(s, \xi) ds. \tag{1.11}$$

$$\widetilde{\phi}(t,\xi) = \frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi) - \int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}\widehat{F}(\tau,\xi)d\tau - \int \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{F}(\tau,\xi)d\tau. \quad (1.12)$$

For the homogeneous part, we rewrite it as

$$\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi) = \frac{e^{it|\xi|}}{2|\xi|}\widehat{\phi}_+ + \frac{e^{-it|\xi|}}{2|\xi|}\widehat{\phi}_-, \quad (1.13)$$

where

$$\widehat{\phi}_\pm = |\xi|\widehat{\phi}_0 \mp i\widehat{\phi}_1. \quad (1.15)$$

Now we split $\widehat{F}$ the same manner as we did to $\widehat{G}$. Let us write

$$\widehat{F}_\pm(\tau,\xi) := h(\pm\tau)\widehat{a}(\tau \mp |\xi|)\widehat{F}(\tau,\xi), \quad (1.16a)$$

$$\widehat{F}_f(\tau,\xi) := \widehat{F}(\tau,\xi) - \big(\widehat{F}_+(\tau,\xi)+\widehat{F}_-(\tau,\xi)\big), \quad (1.16b)$$

For the inhomogeneous part, we obtain

$$\int \Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)} + \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}\Big]\widehat{F}_f(\tau,\xi)d\tau = \int e^{it\tau}\frac{\widehat{F}_f}{|\xi|^2-\tau^2}\,d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int \frac{\widehat{F}_f}{|\xi|-\tau}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{\widehat{F}_f}{|\xi|+\tau}d\tau, \quad (1.17a)$$

$$\int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{it|\xi|}}{2|\xi|}\int \frac{e^{it(\tau-|\xi|)}-1}{|\xi|-\tau}(\widehat{F}_+ + \widehat{a}_6\widehat{F}_-)d\tau + \int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_-}{2|\xi|(|\xi|-\tau)}d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int \frac{(1-\widehat{a}_6)\widehat{F}_-}{|\xi|-\tau}d\tau, \quad (1.17b)$$

$$\int \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{e^{it(\tau+|\xi|)}-1}{|\xi|+\tau}(\widehat{a}_6\widehat{F}_+ + \widehat{F}_-)d\tau + \int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_+}{2|\xi|(|\xi|+\tau)}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{(1-\widehat{a}_6)\widehat{F}_+}{|\xi|+\tau}d\tau, \quad (1.17c)$$

where $\widehat{a}_6(\tau) = \widehat{a}(\frac{\tau}{6})$ and $\widehat{a}$ is the cut-off function defined previously.

Combining (1.17a)-(1.17c), we can give a formula for $\widehat{\phi}$, namely

$$\widehat{\phi}(\tau,\xi)=\sum_{k=0}^{\infty}\Big(\delta_{+}^{(k)}(\tau,\xi)\widehat{B}_{+,k}(\xi)+\delta_{-}^{(k)}(\tau,\xi)\widehat{B}_{-,k}(\xi)\Big)+\widehat{L}(\tau,\xi), \tag{1.18}$$

where $\delta_{\pm}(\tau,\xi)$ are the delta functions supported on $\{\tau=\pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{L}(\tau,\xi):=\frac{\widehat{F}_f}{\widehat{W}(\tau,\xi)}-\frac{(1-\widehat{a}_6(\tau))\widehat{F}_-}{2|\xi|(|\xi|-\tau)}-\frac{(1-\widehat{a}_6(\tau))\widehat{F}_+}{2|\xi|(|\xi|+\tau)}, \tag{1.19a}$$

$$\widehat{B}_{\pm,0}(\xi):=\frac{1}{2|\xi|}\Big[\widehat{\phi}_{\pm}+\int\frac{\widehat{F}_f+(1-\widehat{a}_6(\lambda))\widehat{F}_{\mp}}{|\xi|\mp\lambda}d\lambda\Big], \tag{1.19b}$$

$$\widehat{B}_{\pm,k}(\xi):=\frac{\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{F}_{\pm}+\widehat{a}_6(\lambda)\widehat{F}_{\mp}\big]d\lambda. \tag{1.19c}$$

Remark. We need to localize the solutions for Dirac equation and wave equation due to the presence of the delta function.

2. Estimates.

To localize the solution in time, let $b(t)$ be a cut-off function such that $b(t)$ equals 1 if $|t|\le\frac{1}{2}$, and equals 0 if $|t|>1$, and $b_T(t)=b(t/T)$. For an arbitrary function $f(t,x)$, we have

$$\|\widehat{b}_T*\widehat{f}\|_{L^2}=\|b_Tf\|_{L^2}\le\|b_T\|_{L^\infty}\|f\|_{L^2}. \tag{2.1}$$

For the Dirac equation (1.2), we have the following estimate.

Lemma 2.1. *Let* $0\le\epsilon_1<\frac{1}{2}$, $\alpha_1\ge0$, *and* $\psi_0\in H^{\alpha_1}$. *Then we have*

$$\Big\|\widehat{b}_T*[\widehat{M}^{\alpha_1}\widehat{S}^{1-\epsilon_1}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^3)}\le C\Big(\|\psi_0\|_{H^{\alpha_1}}+\Big\|\frac{\widehat{M}^{\alpha_1}\widehat{G}}{\widehat{S}^{\epsilon_1}}\Big\|_{L^2}\Big). \tag{2.2}$$

Proof. For simplicity, let $\epsilon_1=\epsilon$. Without loss of generality, we prove the special case.

$$\Big\|\widehat{b}_T*[\widehat{S}^{1-\epsilon}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^3)}\le C\Big(\|\psi_0\|_{L^2}+\Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2}\Big). \tag{2.3}$$

To estimate $\widehat{b}_T * [\widehat{S}^{1-\epsilon}\widehat{\psi}]$, we apply formulae (1.8) and (1.9)s. First we compute

$$\begin{aligned}\|\widehat{b}_T * [\widehat{S}^{1-\epsilon}\widehat{K}]\|_{L^2} \le C\|\widehat{S}^{1-\epsilon}\widehat{K}\|_{L^2} \le C\Big\|\widehat{S}^{1-\epsilon}\frac{\widehat{D}}{\widehat{W}}\widehat{G}_f\Big\|_{L^2} + \\ C\Big\|\widehat{S}^{1-\epsilon}\frac{(1-\widehat{a}_6)\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}\Big\|_{L^2} + C\Big\|\widehat{S}^{1-\epsilon}\frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}\Big\|_{L^2} \\ \le C\Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2}. \qquad (2.4)\end{aligned}$$

For the term $\widehat{b}_T * [\widehat{S}^{1-\epsilon}\delta_+^{(k)}\widehat{A}_{+,k}]$, we can mollify $\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k\widehat{S}(\pm|\xi|,\xi) = 0$ if $k \ge 1$. Thus we can compute

$$\begin{aligned}&\|\widehat{b}_T * [\widehat{S}^{1-\epsilon}\delta_+^{(k)}](\xi)\|_{L^2(d\tau)}^2 \\ \sim&\int\Big(\int\widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{1-\epsilon}\delta^{(k)}(\lambda-|\xi|)d\lambda\Big)^2 d\tau \\ \sim&\int\Big(\frac{\partial^k}{\partial\lambda^k}\big(\widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{1-\epsilon}\big)\Big|_{\lambda=|\xi|}\Big)^2 d\tau \\ \le&\int\Big(T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big)^2 d\tau \le T^{2k+1}\|t^k b\|_{L^2} \le CT^{2k+1}. \qquad (2.5)\end{aligned}$$

Then we calculate

$$\begin{aligned}\|\widehat{A}_{+,0}\|_{L^2} \le&\|\psi_0\|_{L^2} + \Big(\int\Big(\int\frac{\widehat{G}_f + \big(1-\widehat{a}_6(\tau)\big)\widehat{G}_-}{\tau-|\xi|}d\tau\Big)^2 d\xi\Big)^{\frac{1}{2}} \\ \le&\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2} \qquad (2.6)\end{aligned}$$

and

$$\begin{aligned}\|\widehat{A}_{+,k}\|_{L^2} \le&\frac{1}{k!}\Big(\int\Big(\int(\tau-|\xi|)^{k-1}\big[\widehat{G}_+ + \widehat{a}_6\widehat{G}_-\big](\tau,\xi)d\tau\Big)^2 d\xi\Big)^{\frac{1}{2}} \\ \le&\frac{2^k}{k!}\Big(\int\int\Big|\widehat{G}_+ + \widehat{a}_6\widehat{G}_-\Big|^2(\tau,\xi)d\tau d\xi\Big)^{\frac{1}{2}} \le \frac{2^k}{k!}\Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2}. \qquad (2.7)\end{aligned}$$

Therefore we have

$$\begin{aligned}\|\widehat{b}_T * [\widehat{S}^{1-\epsilon}\delta_+\widehat{A}_{+,0}]\|_{L^2} \le&T^{\frac{1}{2}}\Big(\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2}\Big), \\ \|\widehat{b}_T * [\widehat{S}^{1-\epsilon}\delta_+^{(k)}\widehat{A}_{+,k}]\|_{L^2} \le&T^{k+\frac{1}{2}}\frac{2^k}{k!}\Big\|\frac{\widehat{G}}{\widehat{S}^{\epsilon}}\Big\|_{L^2}. \qquad (2.8)\end{aligned}$$

The calculation for the term $\widehat{b}_T * [\widehat{S}^{1-\epsilon}\delta_-^{(k)}\widehat{A}_{-,k}]$ is analogous. Combine the above results we complete the proof. □

Consider

$$\begin{cases} \Box I\phi = \mathcal{D}(\mathcal{D}\phi) = I\overline{\psi}\psi, \\ \mathcal{D}\phi(0) = -i\gamma^0\phi_1 - i\gamma^j\partial_j\phi_0. \end{cases} \tag{2.9}$$

For the above equation, we have the following estimate and its proof is analogous to that of Lemma 2.1.

Corollary 2.1. *Let $\delta > 0$. Then we have*

$$\left\|\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\mathcal{D}\phi}\right\|_{L^2} \leq \|\mathcal{D}\phi(0)\|_{L^2} + \left\|\frac{\widehat{\overline{\psi}\psi}}{\widehat{S}^{\frac{1-2\delta}{2}}}\right\|_{L^2}. \tag{2.10}$$

Consider two Dirac equations,

$$\begin{cases} \mathcal{D}\psi_j = G_j, \quad j = 1, 2, \\ \psi_j(0) = \psi_{0j}. \end{cases} \tag{2.11}$$

We have the key estimate whose proof will be presented in the last section.

Lemma 2.2. *(Null Form Estimate) Let $\delta > 0$, and ψ_1, ψ_2 be the solutions for (2.11). If $\psi_{0j} \in H^\alpha$, we have*

$$\begin{aligned} \left\|\frac{\widehat{E}^\delta\widehat{\mathcal{T}(\overline{\psi}_1\psi_2)}}{\widehat{S}^{\frac{1}{2}}}\right\|_{L^2} \leq C(T)\Big(\|\psi_{01}\|_{H^\alpha} + \left\|\frac{\widehat{M}^\alpha\widehat{G}_1}{\widehat{S}^{\frac{\delta}{2}}}\right\|_{L^2}\Big)\cdot \\ \Big(\|\psi_{02}\|_{H^\alpha} + \left\|\frac{\widehat{M}^\alpha\widehat{G}_2}{\widehat{S}^{\frac{\delta}{2}}}\right\|_{L^2}\Big), \end{aligned} \tag{2.12}$$

where $\mathcal{T}(\overline{\psi}\psi)$ is a localization of $\overline{\psi}\psi$.

The Fourier transform of the quadratic expression $\mathcal{T}(\overline{\psi}\psi)$ is written as the sum of the following terms.

$$\big(\delta_\mp\widehat{\overline{A}}_{\pm,0}\big) * \big(\delta_\pm\widehat{A}_{\pm,0}\big) + \sum_{k+l>0}\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\pm^{(l)}\widehat{A}_{\pm,l}\big), \tag{2.13a}$$

$$\big(\delta_\mp\widehat{\overline{A}}_{\pm,0}\big) * \big(\delta_\mp\widehat{A}_{\mp,0}\big) + \sum_{k+l>0}\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big), \tag{2.13b}$$

$$\sum_k\widehat{b}_T * \Big(\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_1 + \widehat{K}_2\big) + \big(\widehat{\overline{K}}_1 + \widehat{\overline{K}}_2\big) * \big(\delta_\pm^{(k)}\widehat{A}_{\pm,k}\big)\Big), \tag{2.13c}$$

$$\widehat{\overline{K}}_1 * \widehat{K}_1 + \widehat{\overline{K}}_1 * \widehat{K}_2 + \widehat{\overline{K}}_2 * \widehat{K}_1 + \widehat{\overline{K}}_2 * \widehat{K}_2. \tag{2.13d}$$

For the wave equation (1.10), we have the following estimate.

Lemma 2.3. *Let ϵ_2 be any real number, $\epsilon_3 > 0$ and ϕ be the solution of (1.10). If $\phi_0 \in H^{1+\epsilon_2}$ and $\phi_1 \in H^{\epsilon_2}$, then*

$$\Big\|\widehat{b}_T * \Big[\widehat{E}^{1+\epsilon_2}\widehat{S}^{\frac{1}{2}+\epsilon_3}\widehat{\phi}\Big]\Big\|_{L^2} \le C\Big(\|\phi_0\|_{H^{1+\epsilon_2}} + \|\phi_1\|_{H^{\epsilon_2}} + \Big\|\frac{\widehat{E}^{\epsilon_2}\widehat{F}}{\widehat{S}^{\frac{1}{2}-\epsilon_3}}\Big\|_{L^2}\Big). \tag{2.14}$$

Proof. For simplicity, we consider the case when $\epsilon_1 = \epsilon_2 = \epsilon$. To estimate $\widehat{b}_T * \big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\widehat{\phi}\big]$ in the L^2-norm, we invoke the formulae (1.19). First we compute

$$\|\widehat{b}_T * \big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\widehat{L}\big]\|_{L^2} \le \|\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\widehat{L}\|_{L^2} \le \Big\|\frac{\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\widehat{F}_f}{\widehat{W}}\Big\|_{L^2} + \Big\|\frac{\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(1-\widehat{a}_6)\widehat{F}_-}{2|\xi|(|\xi|-\tau)}\Big\|_{L^2} + \Big\|\frac{\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(1-\widehat{a}_6)\widehat{F}_+}{2|\xi|(|\xi|+\tau)}\Big\|_{L^2} \le \Big\|\frac{\widehat{E}^{\epsilon}\widehat{F}}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|_{L^2}. \tag{2.15}$$

For the term $\widehat{b}_T * \big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_+^{(k)}\widehat{B}_{+,k}\big]$, we can mollify $\widehat{E}\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k\widehat{S}(\pm|\xi|,\xi) = 0$ if $k \ge 1$. Thus we compute

$$\begin{aligned}
&\big\|\widehat{b}_T * \big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_+^{(k)}\big](\xi)\big\|^2_{L^2(d\tau)} \\
=\ & \int\Big|\int \widehat{b}_T(\tau-\lambda)\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(\lambda,\xi)\delta^{(k)}(\lambda-|\xi|)d\lambda\Big|^2 d\tau \\
=\ & \int\Big|\frac{\partial^k}{\partial\lambda^k}\Big(\widehat{b}_T(\tau-\lambda)\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(\lambda,\xi)\Big)\Big|_{\lambda=|\xi|}\Big|^2 d\tau \\
\sim\ & \int\Big|T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big|^2(|\xi|+1)^{2+2\epsilon}d\tau \\
\le\ & T^{2k+1}\|t^k b\|_{L^2}(|\xi|+1)^{2+2\epsilon} \le\ CT^{2k+1}(|\xi|+1)^{2+2\epsilon},
\end{aligned} \tag{2.16a}$$

which implies

$$\|\widehat{b}_T * \big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_+^{(k)}\widehat{B}_{+,k}\big]\|_{L^2} \le CT^{k+\frac{1}{2}}\Big(\int(|\xi|+1)^{2+2\epsilon}\big|\widehat{B}_{+,k}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}}. \tag{2.16b}$$

To estimate the above integral, we first focus on the region where $|\xi| > 1$.

We have the following bounds:

$$\begin{aligned}&\int\Big|\int\frac{\widehat{M}^{\epsilon}\widehat{F}_f(\lambda,\xi)}{\big||\xi|-\lambda\big|}d\lambda\Big|^2d\xi\\ &\leq\int\int\frac{1}{\big||\xi|-\lambda\big|^{1+2\epsilon}}d\lambda\int\frac{\Big|\widehat{M}^{\epsilon}\widehat{F}_f(\lambda,\xi)\Big|^2}{\big||\xi|-\lambda\big|^{1-2\epsilon}}d\lambda\,d\xi\leq C\Big\|\frac{\widehat{E}^{\epsilon}\widehat{F}_f}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|^2_{L^2}\end{aligned}\tag{2.17}$$

and in the same vein

$$\int\Big|\int\frac{\widehat{M}^{\epsilon}\big(1-\widehat{a}_6(\lambda)\big)\widehat{F}_-(\lambda,\xi)}{\big||\xi|-\lambda\big|}d\lambda\Big|^2d\xi\leq C\Big\|\frac{\widehat{E}^{\epsilon}\widehat{F}_-}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|^2_{L^2}.\tag{2.18}$$

Hence we get

$$\begin{aligned}&\big\|\widehat{b}_T*\big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_+\widehat{B}_{+,0}\big]\big\|_{L^2(L^2(|\xi|>1))}\\ &\leq CT^{\frac{1}{2}}\Big(\|\phi_0\|_{H^{1+\epsilon}}+\|\phi_1\|_{H^{\epsilon}}+\Big\|\frac{\widehat{E}^{\epsilon}\widehat{F}}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|_{L^2}\Big)\end{aligned}\tag{2.19}$$

and

$$\begin{aligned}&\big\|\widehat{b}_T*\big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_+^{(k)}\widehat{B}_{+,k}\big]\big\|_{L^2(L^2(|\xi|>1))}\\ &\leq CT^{k+\frac{1}{2}}\frac{c^k}{k!}\Big(\int\int\widehat{E}^{2\epsilon}\big|\widehat{F}_++\widehat{a}_6\widehat{F}_-\big|^2(\tau,\xi)d\lambda\,d\xi\Big)^{\frac{1}{2}}\\ &\leq CT^{k+\frac{1}{2}}\frac{c^k}{k!}\Big\|\frac{\widehat{E}^{\epsilon}\widehat{F}}{\widehat{S}^{\frac{1}{2}-\epsilon}}\Big\|_{L^2}.\end{aligned}\tag{2.20}$$

The calculation for the term $\widehat{b}_T*\big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}\delta_-^{(k)}\widehat{B}_{-,k}\big]$ is analogous.

For region $|\xi|\leq 1$, we consider $\widehat{b}_T*\big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(\delta_+^{(k)}\widehat{B}_{+,k}+\delta_-^{(k)}\widehat{B}_{-,k})\big]$. This is clear from the derivation of the solution representation which indicates that the solution is actually not singular along the cones.

$$\begin{aligned}&\widehat{b}_T*\big[\widehat{E}^{1+\epsilon}\widehat{S}^{\frac{1}{2}+\epsilon}(\delta_+^{(k)}\widehat{B}_{+,k}+\delta_-^{(k)}\widehat{B}_{-,k})\big](\tau,\xi)\\ &\sim T^{k+1}\big(|\xi|+1\big)^{1+\epsilon}\big[\widehat{t^kb}\big(T(\tau-|\xi|)\big)\widehat{B}_{+,k}(\xi)+\widehat{t^kb}\big(T(\tau+|\xi|)\big)\widehat{B}_{-,k}(\xi)\big]\\ &=T^{k+1}\big(|\xi|+1\big)^{1+\epsilon}\big[\widehat{t^kb}\big(T(\tau-|\xi|)\big)-\widehat{t^kb}\big(T(\tau+|\xi|)\big)\big]\widehat{B}_{+,k}(\xi)+\\ &\quad T^{k+1}\big(|\xi|+1\big)^{1+\epsilon}\widehat{t^kb}\big(T(\tau+|\xi|)\big)\big[\widehat{B}_{+,k}(\xi)+\widehat{B}_{-,k}(\xi)\big].\end{aligned}\tag{2.21}$$

Under the restriction of $|\xi| \leq 1$, we have

$$\widehat{t^k b}\big(T(\tau-|\xi|)\big)-\widehat{t^k b}\big(T(\tau+|\xi|)\big)\sim T\widehat{t^{k+1}b}\big(T(\tau-(1-2\theta)|\xi|)\big)|\xi|, \tag{2.22}$$

$$\widehat{B}_{+,0}(\xi)+\widehat{B}_{-,0}(\xi)\sim\widehat{\phi}_0+\int\frac{\widehat{F}_f}{|\xi|^2-\lambda^2}\,d\lambda, \tag{2.23}$$

and

$$\widehat{B}_{+,k}(\xi)+\widehat{B}_{-,k}(\xi)\sim\frac{1}{(k-1)!}\int\big(\lambda-(1-2\theta)|\xi|\big)^{k-2}(\widehat{F}_+ +\widehat{a}_6\widehat{F}_-)\,d\lambda. \tag{2.24}$$

Combine the above results we complete the proof. □

Since a Dirac equation implies a wave equation, thus we consider

$$\begin{cases}\Box\psi=\mathcal{D}(\mathcal{D}\psi)=(\mathcal{D}\phi)\psi+\phi^2\psi,\\ \psi(0)=\psi_0,\quad \partial_t\psi(0)=i\gamma^0(i\gamma^j\partial_j\psi_0+\phi_0\psi_0).\end{cases} \tag{2.25a}$$

We have the following estimate whose proof is analogous to that of Lemma 2.3.

Corollary 2.3. *Let $\delta>0$. Then we have*

$$\big\|\widehat{E}^\mu\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi}\big\|_{L^2}\leq C(T)\Big(\|\psi_0\|_{H^\mu}+\|\phi_0\|_{H^1}\|\psi_0\|_{H^\mu}+\Big\|\frac{\widehat{(\mathcal{D}\phi)\psi}+\widehat{\phi^2\psi}}{\widehat{E}^{1-\mu}\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2}\Big). \tag{2.25b}$$

We will also need some technical lemmas.

Lemma 2.4. *(Hardy-Littlewood-Polya) Let $r=2-\frac{1}{p}-\frac{1}{q}$. Then we have*

$$\int_{\mathbb{R}^1\times\mathbb{R}^1}\frac{f(s)g(t)}{|s-t|^r}dsdt\leq C\|f\|_{L^p}\|g\|_{L^q}. \tag{2.26}$$

Lemma 2.5. *Let $f(t,x)$ be any function. Assume that $\epsilon_1\geq 0$, $r_1=\frac{2}{1+\epsilon_1}$, $0\leq\epsilon_2\leq\frac{1}{2}$, and $r_2=\frac{2}{1-2\epsilon_2}$. Then we have*

$$\Big\|\frac{\widehat{b_T f}}{\widehat{S}^{\epsilon_1}}\Big\|_{L^2}\leq C\|b_T f\|_{L^{r_1}(L^2)}, \tag{2.27a}$$

$$\|f\|_{L^{r_2}([0,T];L^2)}\leq C\|\widehat{S}^{\epsilon_2}\widehat{f}\|_{L^2}. \tag{2.27b}$$

Proof. They are dual estimates if $\epsilon_1 = 2\epsilon_2$, hence we only show (2.27a). For simplicity, let $\epsilon_1 = \epsilon$. Through some analysis, we have

$$\mathcal{F}_\tau^{-1}\big(\widehat{S}^{-\epsilon}\big)(t,\xi) \sim \frac{e^{\pm it|\xi|}}{|t|^{1-\frac{\epsilon}{2}}}, \tag{2.28}$$

which implies

$$\mathcal{F}_\tau^{-1}\Big(\frac{\widehat{b_T f}}{\widehat{S}^\epsilon}\Big)(t,\xi) \sim \int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\frac{\epsilon}{2}}}(\widetilde{b_T f})(s,\xi)ds. \tag{2.29}$$

Invoke (2.29), duality argument, and Hardy-Littlewood-Polya inequality, then we can compute

$$\begin{aligned}
&\Big|\Big\langle b_T f, g\Big\rangle\Big| = \Big|\Big\langle \frac{\widehat{b_T f}}{\widehat{S}^\epsilon}, \widehat{S}^\epsilon \widehat{g}\Big\rangle\Big| \\
=&\Big|\iint \int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\frac{\epsilon}{2}}}(\widetilde{b_T f})(s,\xi)ds\overline{\mathcal{F}_\tau^{-1}(\widehat{S}^\epsilon\widehat{g})(t,\xi)}\, dtd\xi\Big| \\
\leq&\int \frac{\|(\widetilde{b_T f})(s)\|_{L^2}\|\mathcal{F}_\tau^{-1}(\widehat{S}^\epsilon\widehat{g})(t)\|_{L^2}}{|t-s|^{1-\frac{\epsilon}{2}}}dsdt \\
\leq&C\|\widetilde{b_T f}\|_{L^r(L^2)}\|\mathcal{F}_\tau^{-1}(\widehat{S}^\epsilon\widehat{g})\|_{L^2(L^2)} = C\|b_T f\|_{L^r(L^2)}\|\widehat{S}^\epsilon\widehat{g}\|_{L^2}.
\end{aligned} \tag{2.30}$$

This completes the proof of (2.27a). □

Lemma 2.6. *Assume that* $0 < \gamma < 1$, $0 < \beta < \dfrac{1}{2}$, $0 < 2\beta - \gamma < 1$, $p = \dfrac{2}{1-\gamma}$, *and* $r = \dfrac{2}{1+\gamma-2\beta}$. *Then we have*

$$\|f\|_{L^r([0,T];L^p)} \leq C\|\widehat{E}^\gamma\widehat{S}^\beta\widehat{f}\|_{L^2}, \tag{2.31a}$$

and

$$\|\widehat{E}^{-\gamma}\widehat{S}^{-\beta}\widehat{g}\|_{L^2} \leq C\|g\|_{L^{r'}([0,T];L^{p'})}, \tag{2.31b}$$

where r' *and* p' *are dual exponents of* r *and* p *respectively.*

Proof. Since the two estimates are dual, hance we only show (2.31a) Let g be an arbitrary function. First, we compute

$$\begin{aligned}
|\langle f, g\rangle| =&\big|\langle \widehat{E}^\gamma\widehat{S}^\beta\widehat{f}, \widehat{E}^{-\gamma}\widehat{S}^{-\beta}\widehat{g}\rangle\big| \\
\leq&\Big\|\widehat{E}^\gamma\widehat{S}^\beta\widehat{f}\Big\|_{L^2}\Big|\langle\widehat{g}, \widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{g}\rangle\Big|^{\frac{1}{2}}.
\end{aligned} \tag{2.32}$$

To prove the estimate (2.31), we employ the dyadic (Paley-Littlewood) decomposition for the function $g(t,x)$. Let us introduce an auxiliary function $\widehat{\beta}(\xi)$ which is supported in the region $\{\frac{1}{2} < |\xi| < 2\}$. Denote

$$\widehat{\beta}_k(\xi) := \widehat{\beta}(\frac{\xi}{2^k}),\ \widehat{\eta} = 1 - \sum_{k=1}^{\infty} \widehat{\beta}_k, \quad \widehat{\varphi} := \widehat{\beta}_{-1} + \widehat{\beta}_0 + \widehat{\beta}_1, \tag{2.33a}$$

$$\widehat{g}_0 := \widehat{\eta}\widehat{g}, \qquad\qquad \widehat{\varphi}_0(\xi) := \widehat{\eta}(\frac{\xi}{2})\,, \tag{2.33b}$$

$$\widehat{g}_k := \widehat{\beta}_k(\xi)\widehat{g}, \quad \widehat{\varphi}_k(\xi) := \widehat{\varphi}(\frac{\xi}{2^k}), \quad k = 1, 2, 3, \cdots. \tag{2.33c}$$

Notice that $\widehat{g}_k(\tau,\xi) = \widehat{\varphi}_k(\xi)\widehat{g}_k(\tau,\xi)$. Through some analysis, we have

$$\mathcal{F}_\tau^{-1}\big(\widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\big)(t,\xi) \sim \frac{C}{|t|^{1-2\beta}} \frac{e^{\pm it|\xi|}}{(|\xi|+1)^{2\gamma}} \ell(t), \tag{2.34a}$$

where $|\ell(t)| \leq C$, and a classic estimate gives

$$\Big|\mathcal{F}_\xi^{-1}(e^{\pm it|\xi|}\widehat{\varphi})(t,x)\Big| \leq \frac{C(\varphi)}{|t|} \tag{2.34b}$$

Let us write

$$K(t)g_k(x) = \mathcal{F}^{-1}\Big(\widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{g}_k\Big)(t,x). \tag{2.35}$$

Thus the estimates (2.34a) and (2.34b) imply, for $k = 0, 1, 2, \cdots$,

$$\|K(t)g_k\|_{L^\infty} \leq \int \frac{2^{2(1-\gamma)k}}{|t-s|^{2(1-\beta)}} \|g_k(s)\|_{L^1}\, ds, \tag{2.36}$$

$$\|K(t)g_k\|_{L^2} \leq \int \frac{2^{-2\gamma k}}{|t-s|^{1-2\beta}} \|g_k(s)\|_{L^2}\, ds. \tag{2.37}$$

The proof of (2.37) is straight forward and we will show (2.36) at the end of the paper. Assuming the validity of the above two estimates, the interpolation between them gives

$$\|K(t)g_k\|_{L^p} \leq \int \frac{C}{|t-s|^{1+\gamma-2\beta}} \|g_k(s)\|_{L^{p'}}\, ds, \tag{2.38}$$

where $p = \dfrac{2}{1-\gamma}$.

Now we invoke (2.38) and (2.26) to compute

$$\begin{aligned}
&\left|\langle \widehat{g}_k, \widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{g}_k\rangle\right| \\
=&\left| \int g_k(t,x)\overline{\mathcal{F}^{-1}\left(\widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{g}_k\right)}(t,x)dtdx\right| \\
\leq&\left| \int \|g_k(t)\|_{L^{p'}}\|K(t)g_k\|_{L^p}dt\right| \\
\leq&\int \frac{\|g_k(t)\|_{L^{p'}}\|g_k(s)\|_{L^{p'}}}{|t-s|^{1+\gamma-2\beta}}dsdt \leq \|g_k\|^2_{L^{r'}(L^{p'})}. \qquad (2.39)
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
|\langle f, g\rangle| \leq&\left\|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{f}\right\|_{L^2}\Big(\sum_{k=0}^{\infty}\left|\langle \widehat{g}_k, \widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{g}_k\rangle\right|\Big)^{\frac{1}{2}} \\
\leq&\left\|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{f}\right\|_{L^2}\Big(\sum_{k=0}^{\infty}\|g_k\|^2_{L^{r'}(L^{p'})}\Big)^{\frac{1}{2}} \\
\leq&\left\|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{f}\right\|_{L^2}\|g\|_{L^{r'}(B^0_{p',2})}. \qquad (2.40)
\end{aligned}$$

This completes the proof. □

Remark. Through an analogous argument, we can actually obtain a better estimate

$$\|f\|_{L^r([0,T];L^p)} \leq C\|\widehat{M}^{\gamma}\widehat{S}^{\beta}\widehat{f}\|_{L^2},$$

where r, p, γ, and β satisfy the same conditions as those in Lemma 2.6.

3. Local Existence.

Now we are ready to prove the local existence for the (DKG) equations.

Proof of Theorem 0.1. Consider a Picard iteration scheme for the Dirac equation

$$\mathcal{D}\psi^{k+1} = b_T\phi^k\psi^k; \qquad (3.1a)$$

$$\psi^{k+1}(0,x) := \psi_0(x), \qquad (3.1b)$$

and for the Klein-Gordon equation

$$\Box\phi^{k+1} = \mathcal{T}(\overline{\psi}^{k+1}\psi^{k+1}); \tag{3.1c}$$

$$\phi^{k+1}(0,x) = \phi_0(x), \quad \phi^{k+1}_{,t}(0,x) = \phi_1(x), \tag{3.1d}$$

where $\mathcal{T}(\overline{\psi}\psi)$ is defined in (2.13a-d), which is a localization of $\overline{\psi}\psi$.

Iteration scheme introduces a map $\mathcal{T}_1$ defined by

$$\mathcal{T}_1(\phi^k, \psi^k) = (\phi^{k+1}, \psi^{k+1}). \tag{3.2a}$$

We want to show that $\mathcal{T}_1$ is a contraction under the norm

$$\mathcal{N}(\phi,\psi) = \left\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi}\right\|_{L^2} + \left\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi}\right\|_{L^2} \tag{3.2b}$$

and the condition $0 < \delta < \epsilon$. For convenience, we call

$$J(0) = \|\phi_0\|_{H^1} + \|\phi_1\|_{L^2} + \|\psi_0\|^2_{H^\mu} + 1. \tag{3.3}$$

First we employ (2.31a) to obtain the following estimates.

$$\|\partial^\alpha\phi\|_{L^{r_1}(L^{p_1})} \le \|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{M}^{\alpha}\widehat{\phi}\|_{L^2} \le \left\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi}\right\|_{L^2}, \tag{3.4a}$$

where $r_1 = \dfrac{8}{3-4\epsilon}$ and $p_1 = \dfrac{8}{1+4\epsilon}$.

$$\|\psi\|_{L^{r_2}(L^{p_2})} \le \|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{\psi}\|_{L^2} \le \left\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi}\right\|_{L^2}, \tag{3.4b}$$

where $r_2 = \dfrac{8}{1+4\epsilon}$ and $p_2 = \dfrac{8}{3-4\epsilon}$.

$$\|\phi\|_{L^{r_3}(L^{p_3})} \le \|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{\phi}\|_{L^2} \le \left\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi}\right\|_{L^2}, \tag{3.4c}$$

where $r_3 = \dfrac{2}{1-\nu}$ and $p_3 = \dfrac{2}{\nu}$.

$$\|\partial^\alpha\psi\|_{L^{r_4}(L^{p_4})} \le \|\widehat{E}^{\gamma}\widehat{S}^{\beta}\widehat{M}^{\alpha}\widehat{\phi}\|_{L^2} \le \left\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi}\right\|_{L^2}, \tag{3.4d}$$

where $r_4 = \dfrac{2}{\nu}$ and $p_4 = \dfrac{2}{1-\nu}$.

we apply (2.14), (2.12), and (2.27) to compute

$$\begin{aligned}\mathcal{N}(\mathcal{T}_1\phi) &= \big\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\mathcal{T}_1\phi}\big\|_{L^2}\\ &\le C\Big(J(0)+\Big\|\frac{\widehat{\mathcal{T}\overline{\psi}\psi}}{\widehat{E}^{\nu}\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2}\Big)\\ &\le C\Big(J(0)+\Big\|\frac{\widehat{M}^{\alpha}\widehat{b_T\phi\psi}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|^2_{L^2}\Big)\\ &\le C\Big(J(0)+\Big\|\widehat{M}^{\alpha}\widetilde{\phi\psi}\Big\|^2_{L^{\frac{4}{2+\delta}}([0,T];L^2)}\Big). \end{aligned} \tag{3.5}$$

To bound the term above, we observe that

$$\widehat{M}(\xi+\eta)\le\widehat{M}(\xi)+\widehat{M}(\eta), \tag{3.6}$$

which implies that

$$\widehat{M}^{\alpha}\widehat{\phi\psi}\le\big(\widehat{M}^{\alpha}\widehat{\phi}\big)*\widehat{\psi}+\widehat{\phi}*\big(\widehat{M}^{\alpha}\widehat{\psi}\big). \tag{3.7}$$

For convenience, we use the notation

$$\partial_x^{\alpha}f:=\mathcal{F}_{\xi}^{-1}\big(\widehat{M}^{\alpha}\widetilde{f}\big). \tag{3.8}$$

Then we invoke (3.4a-d) to obtain

$$\begin{aligned}&\Big\|\widehat{M}^{\alpha}\widetilde{\phi\psi}\Big\|_{L^{\frac{4}{2+\delta}}([0,T];L^2)}\\ \le&\Big\|\big(\partial_x^{\alpha}\phi\big)\psi\Big\|_{L^{\frac{4}{2+\delta}}([0,T];L^2)}+\Big\|\phi\big(\partial_x^{\alpha}\psi\big)\Big\|_{L^{\frac{4}{2+\delta}}([0,T];L^2)}\\ \le&T^{\frac{\delta}{8}}\big\|\partial_x^{\alpha}\phi\big\|_{L^{r_1}(L^{p_1})}\big\|\psi\big\|_{L^{r_2}(L^{p_2})}+T^{\frac{\delta}{4}}\big\|\phi\big\|_{L^{r_3}(L^{p_3})}\big\|\partial_x^{\alpha}\psi\big\|_{L^{r_4}(L^{p_4})}\\ \le&T^{\frac{\delta}{8}}\big\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi}\big\|_{L^2}\big\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi}\big\|_{L^2}. \end{aligned} \tag{3.9}$$

Hence, using (3.5) and (3.9), we have

$$\mathcal{N}(\mathcal{T}_1(\phi))\le C\big(J(0)+T^{\frac{\delta}{4}}\mathcal{N}^4(\psi,\phi)\big). \tag{3.10}$$

For the spinor field ψ, we first invoke (2.27b), (2.10), and (2.12) to derive

$$\begin{aligned}
\|\mathcal{D}\phi\|_{L^\infty([0,T];L^2)} \leq \|\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\mathcal{D}\phi}\|_{L^2} \leq \|\mathcal{D}\phi(0)\|_{L^2} + \Big\|\frac{\widehat{\overline{\psi}\psi}}{\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2} \\
\leq &\|\phi_0\|_{H^1} + \|\phi_1\|_{L^2} + \|\psi_0\|_{H^\alpha}^2 + \Big\|\frac{\widehat{M}^\alpha\widehat{\phi\psi}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}^2 \\
\leq &J(0) + T^{\frac{\delta}{4}}\mathcal{N}^4(\phi,\psi) \qquad (3.11)
\end{aligned}$$

Next we apply (3.4b) and (2.27) to compute

$$\begin{aligned}
\|(\mathcal{D}\phi)\psi\|_{L^{r'}([0,T];L^{p'})} \leq &\|\psi\|_{L^{r_2}(L^{p_2})}\|\mathcal{D}\phi\|_{L^{\frac{2}{1-2\delta}}(L^2)} \\
\leq &T^{\frac{1-2\delta}{2}}\mathcal{N}(\psi)\|\mathcal{D}\phi\|_{L^\infty([0,T];L^2)}
\end{aligned} \qquad (3.12)$$

and

$$\|\phi^2\psi\|_{L^{r'}([0;T];L^{p'})} \leq \|\psi\|_{L^{r_2}(L^{p_2})}\|\phi\|^2_{L^{\frac{4}{1-2\delta}}(L^4)} \leq T^\delta\mathcal{N}^3(\phi,\psi), \qquad (3.13)$$

where $r' = \dfrac{8}{5+4\epsilon-8\delta}$ and $p' = \dfrac{8}{7-4\epsilon}$.

Finally we employ (2.25b) and (2.31b) to get

$$\begin{aligned}
\mathcal{N}(\mathcal{T}_1\psi) = &\big\|\widehat{E}^\mu\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\mathcal{T}_1\psi}\big\|_{L^2} \\
\leq &C(T)\Big(\|\psi_0\|_{H\mu} + \|\phi_0\|_{H^1}\|\psi_0\|_{H^\mu} + \Big\|\frac{\widehat{(\mathcal{D}\phi)\psi} + \widehat{\phi^2\psi}}{\widehat{E}^{1-\mu}\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2}\Big) \\
\leq &J(0) + \|(\mathcal{D}\phi)\psi + \phi^2\psi\|_{L^{r'}([0;T];L^{p'})} \\
\leq &C(T)\big(J(0) + T^\delta\mathcal{N}^5(\phi,\psi)\big) \qquad (3.14)
\end{aligned}$$

Thus we combine the above results to get

$$\mathcal{N}(\mathcal{T}_1(\psi,\phi)) \leq C(T)\big(J(0) + T^{\frac{\delta}{4}}\mathcal{N}^5(\phi,\psi)\big).$$

Choosing sufficiently large L, for suitable T, we have

$$\mathcal{N}(\psi,\phi) \leq L \Longrightarrow \mathcal{N}\big(\mathcal{T}_1(\psi,\phi)\big) \leq L, \qquad (3.15)$$

provided that

$$C(J(0)+T^{\frac{\delta}{4}}L^5)\le L. \tag{3.16}$$

Now we consider the difference $\mathcal{T}_1(\psi,\phi)-\mathcal{T}_1(\psi',\phi')$. Base on the observations

$$2(\overline{\psi}\psi-\overline{\psi'}\psi')=(\overline{\psi-\psi'})(\psi+\psi')+(\overline{\psi+\psi'})(\psi-\psi'), \tag{3.17a}$$

$$2(\phi\psi-\phi'\psi')=(\phi-\phi')(\psi+\psi')+(\phi+\phi')(\psi-\psi'), \tag{3.17b}$$

Employing (2.14), (2.12), and (3.17), we first calculate

$$\begin{aligned}
&\mathcal{N}(\mathcal{T}_1\phi-\mathcal{T}_1\phi')=\big\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\mathcal{F}(\mathcal{T}_1\phi-\mathcal{T}_1\phi')\big\|_{L^2}\\
&\le C\Big(\Big\|\frac{\mathcal{F}(\mathcal{T}(\overline{\psi-\psi'})(\psi+\psi'))}{\widehat{E}^{\nu}\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2}+\Big\|\frac{\mathcal{F}(\mathcal{T}(\overline{\psi+\psi'})(\psi-\psi'))}{\widehat{E}^{\nu}\widehat{S}^{\frac{1-2\delta}{2}}}\Big\|_{L^2}\Big)\\
&\le C\Big(\Big\|\frac{\widehat{M}^{\alpha}\mathcal{F}(b_T(\phi-\phi')(\psi+\psi'))}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}+\Big\|\frac{\widehat{M}^{\alpha}\mathcal{F}(b_T(\phi+\phi')(\psi-\psi'))}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big)\cdot\\
&\qquad\cdot\Big(J(0)+\Big\|\frac{\widehat{M}^{\alpha}\mathcal{F}(b_T(\phi\psi+\phi'\psi'))}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big)\\
&\le CT^{\frac{\delta}{8}}\big(\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi-\psi'}\|_{L^2}\big)L(J(0)+T^{\frac{\delta}{8}}L^2)\\
&\le CT^{\frac{\delta}{8}}L^3\big(\mathcal{N}(\phi-\phi')+\mathcal{N}(\psi-\psi')\big)
\end{aligned} \tag{3.18}$$

Analogously, we get

$$\begin{aligned}
\mathcal{N}(\mathcal{T}_1\psi-\mathcal{T}_1\psi')&=\big\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\mathcal{F}(\mathcal{T}_1\psi-\mathcal{T}_1\psi')\big\|_{L^2}\\
&\le CT^{\delta}L^4\Big(\|\widehat{E}^{1-\nu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{E}^{\mu}\widehat{S}^{\frac{1+2\delta}{2}}\widehat{\psi-\psi'}\|_{L^2}\Big).
\end{aligned} \tag{3.19}$$

Combining (3.18)and (3.19), we have

$$\mathcal{N}\big(\mathcal{T}_1(\psi-\psi',\phi-\phi')\big)\le CT^{\frac{\delta}{8}}L^4N(\psi-\psi',\phi-\phi'). \tag{3.20}$$

Therefore for suitable T, we obtain

$$\mathcal{N}\big(\mathcal{T}_1(\psi-\psi',\phi-\phi')\big)\le\frac{1}{2}N(\psi-\psi',\phi-\phi'), \tag{3.21}$$

provided that

$$CT^{\frac{\delta}{8}}L^4\le\frac{1}{2}. \tag{3.22}$$

We can conclude that the map $\mathcal{T}_1$ is indeed a contraction with respect to the norm $\mathcal{N}$, thus it has a unique fixed point. □

4. Null Form Estimate.

In this section, we demonstrate the proof of the key estimate.

Lemma 2.2. *(Null Form Estimate) Let $\delta > 0$, and ψ_1, ψ_2 be the solutions for (2.11). If the initial data $\psi_{0j} \in H^\alpha$, $j = 1, 2$, then we have*

$$\Big\| \frac{\widehat{E}^\delta \widehat{\mathcal{T}(\overline{\psi}_1 \psi_2)}}{\widehat{S}^{\frac{1}{2}}} \Big\|_{L^2} \leq C(T)\Big(\|\psi_{01}\|_{H^\alpha} + \Big\| \frac{\widehat{M}^\alpha \widehat{G}_1}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2} \Big) \Big(\|\psi_{02}\|_{H^\alpha} + \Big\| \frac{\widehat{M}^\alpha \widehat{G}_2}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2} \Big). \quad (4.1)$$

The proof for the estimate is based on the duality argument and it will be given in a number of steps. Without loss of generality, we assume that $\psi_1 = \psi_2$, and prove the following case:

$$\Big\| \frac{\widehat{E}^\delta \widehat{\mathcal{T}(\overline{\psi} \psi)}}{\widehat{S}^{\frac{1}{2}}} \Big\|_{L^2} \leq C(T)\Big(\|\psi_0\|_{H^\alpha} + \Big\| \frac{\widehat{M}^\alpha \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2} \Big)^2. \quad (4.2)$$

Recall that the notations:

$$\widehat{E}(\tau, \xi) := |\tau| + |\xi| + 1, \qquad \widehat{S}(\tau, \xi) := \Big| |\tau| - |\xi| \Big| + 1, \quad (4.3a)$$

$$\widehat{W}(\tau, \xi) := \tau^2 - |\xi|^2, \qquad \widehat{D}(\tau, \xi) := \gamma^0 \tau + \gamma^1 \xi, \quad (4.3b)$$

$$\widehat{D}_+ := \widehat{D}(|\xi|, +\xi), \qquad \widehat{D}_- := \widehat{D}(|\xi|, -\xi). \quad (4.3c)$$

The formula for $\widehat{\psi}$, as in (1.8), for the Dirac equation (1.2) is given by

$$\widehat{\psi}(\tau, \xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau, \xi) \widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau, \xi) \widehat{A}_{-,k}(\xi) \Big) + \widehat{K}(\tau, \xi), \quad (4.4)$$

where $\delta_\pm(\tau, \xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau, \xi) := \frac{\widehat{D}(\tau, \xi)}{\widehat{W}(\tau, \xi)} \widehat{G}_f + \frac{(1 - \widehat{a}_6(\tau)) \widehat{D}_+ \widehat{G}_-}{2|\xi|(\tau - |\xi|)} + \frac{(1 - \widehat{a}_6) \widehat{D}_- \widehat{G}_+}{2|\xi|(\tau + |\xi|)}, \quad (4.5a)$$

$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|} \Big[\gamma^0 \widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1 - \widehat{a}_6(\lambda)) \widehat{G}_\mp}{\lambda \mp |\xi|} d\lambda \Big], \quad (4.5b)$$

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm (-1)^k}{2|\xi| k!} \int (\lambda \mp |\xi|)^{k-1} \big[\widehat{G}_\pm + \widehat{a}_6(\lambda) \widehat{G}_\mp \big] d\lambda. \quad (4.5c)$$

Moreover we write

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\widehat{f}_{\pm,k}(\xi), \tag{4.6}$$

and split $\widehat{K} = \widehat{K}_1 + \widehat{K}_2$, where

$$\widehat{K}_1 := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f; \quad \widehat{K}_2 := \frac{b_1\widehat{D}_+\widehat{G}_- + b_2\widehat{D}_-\widehat{G}_+}{\widehat{E}\widehat{S}}, \tag{4.7}$$

and b_1, b_2 are bounded functions. $\widehat{\mathcal{T}(\overline{\psi}\psi)}$ is the sum of the following terms.

$$\big(\delta_\mp\widehat{\overline{A}}_{\pm,0}\big) * \big(\delta_\pm\widehat{A}_{\pm,0}\big) + \sum_{k+l>0}\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\pm^{(l)}\widehat{A}_{\pm,l}\big), \tag{4.8a}$$

$$\big(\delta_\mp\widehat{\overline{A}}_{\pm,0}\big) * \big(\delta_\mp\widehat{A}_{\mp,0}\big) + \sum_{k+l>0}\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big), \tag{4.8b}$$

$$\sum_k \widehat{b}_T * \Big(\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big)*\big(\widehat{K}_1 + \widehat{K}_2\big)+\big(\widehat{\overline{K}}_1 + \widehat{\overline{K}}_2\big)*\big(\delta_\pm^{(k)}\widehat{A}_{\pm,k}\big)\Big), \tag{4.8c}$$

$$\widehat{\overline{K}}_1 * \widehat{K}_1 + \widehat{\overline{K}}_1 * \widehat{K}_2 + \widehat{\overline{K}}_2 * \widehat{K}_1 + \widehat{\overline{K}}_2 * \widehat{K}_2. \tag{4.8d}$$

Notice that

$$\widehat{A^\dagger_{\pm,k}}(\xi) = \widehat{A}^\dagger_{\pm,k}(-\xi); \qquad \widehat{f^{+}_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi), \tag{4.9a}$$

$$\widehat{\overline{A}_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi)\frac{\widehat{D}_\pm}{|\xi|}\gamma^0; \quad \widehat{\overline{K}}(\tau,\xi) = \widehat{K}^\dagger(-\tau,-\xi)\gamma^0, \tag{4.9b}$$

and

$$\widehat{\overline{\psi}}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_-^{(k)}(\tau,\xi)\widehat{\overline{A}}_{+,k}(\xi) + \delta_+^{(k)}(\tau,\xi)\widehat{\overline{A}}_{-,k}(\xi)\Big) + \widehat{\overline{K}}(\tau,\xi), \tag{4.10}$$

Lemma 4.1. *Let $k+l>0$ and $\delta \geq 0$. The following estimates hold*

$$\Big\|\frac{\widehat{E}^{2\delta}\big(\delta_\mp\widehat{\overline{A}}_{\pm,0}\big) * \big(\delta_\mp\widehat{A}_{\mp,0}\big)}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \leq C\|f_{\pm,0}\|_{H^\alpha}\|f_{\mp,0}\|_{H^\alpha}, \tag{4.11a}$$

$$\Big\|\frac{\widehat{E}^{2\delta}\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big)}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \\ \leq C(k+l)T^{k+l-\frac{1}{2}}\|f_{\pm,k}\|_{H^\alpha}\|f_{\mp,l}\|_{H^\alpha}. \tag{4.11b}$$

Proof. We only demonstrate the case (4.11b) since the proof for (4.11a) is analogous. Let us call

$$\widehat{Z}_{\pm,k} \equiv \delta_{\pm}^{(k)} \widehat{A}_{\pm,k} = \delta_{\pm}^{(k)} \frac{\widehat{D}_{\pm}}{2|\xi|} \widehat{f}_{\pm,k}. \tag{4.12}$$

Using duality, we demonstrate the case $(-,+)$, while the case $(+,-)$ is being similar. We first compute the following term

$$\widehat{D}_{-}(\xi)\gamma^0\widehat{D}_{+}(\eta) = \gamma^0(|\xi||\eta| - \xi\cdot\eta) + \gamma^0\gamma^j\gamma^k\xi_j\eta_k\Big|_{j\neq k} - \gamma^j(|\eta|\xi_j - |\xi|\eta_j). \tag{4.13}$$

Thus

$$\begin{aligned} &\Big|\left\langle b_T\overline{Z}_{-,k}Z_{+,l},g\right\rangle\Big| \\ =&\Big|\int \widehat{f}^{\dagger}_{-,k}(-\xi)\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}b_T}g}(|\xi|+|\eta|,\xi+\eta)\,d\xi d\eta\Big| \\ \leq&\|f_{-,k}\|_{H^\alpha}\|f_{+,l}\|_{H^\alpha}\Big(\int\frac{|\xi||\eta|-\xi\cdot\eta}{|\xi|^{1+2\alpha}|\eta|^{1+2\alpha}}\big|\widehat{t^{k+l}b_T}g(|\xi|+|\eta|,\xi+\eta)\big|^2\,d\xi d\eta\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.14}$$

To estimate the above integral, we change the variables by

$$|\xi|+|\eta|=\tau, \quad \xi+\eta=z, \quad \xi=\rho\omega, \quad |\omega|=1, \tag{4.15}$$

thus we can rewrite it as

$$\int E(\tau,z)\big|\widehat{t^{k+l}b_T}g(\tau,z)\big|^2\,d\tau dz, \tag{4.16}$$

where

$$E(\tau,z) = \int \frac{|\xi||\eta|-\xi\cdot\eta}{|\xi|^{1+2\alpha}|\eta|^{1+2\alpha}}\rho^2\frac{d\rho}{d\tau}d\omega. \tag{4.17}$$

Throughout some computations, we have

$$E(\tau,z) \leq C\frac{\tau-|z|}{\tau^{4\delta}}, \tag{4.18}$$

$$\big\|(|\tau|+1)^{\frac{1}{2}}\widehat{t^{k+l}b_T}\big\|_{L^1} \leq C(k+l)T^{k+l-\frac{1}{2}}\|b\|_{H^1}. \tag{4.19}$$

With the aid of the above inequalities and the observation

$$\frac{\big||\tau|-|\xi|\big|+1}{(|\tau|+|\xi|+1)^{4\delta}} \leq C\frac{\big||\tau-\sigma|-|\xi|\big|+1}{(|\tau-\sigma|+|\xi|+1)^{4\delta}}\big(|\sigma|+1\big)^{1-4\delta}, \tag{4.20}$$

we can estimate

$$\begin{aligned}\|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}}\widehat{t^{k+l}b_T}g\|_{L^2} \leq & \big\|(|\tau|+1)^{\frac{1}{2}}\widehat{t^{k+l}b_T}\big\|_{L^1}\Big\|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}}\widehat{g}\Big\|_{L^2} \\ \leq & C(k+l)T^{k+l-\frac{1}{2}}\Big\|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}}\widehat{g}\Big\|_{L^2}. \end{aligned}\tag{4.21}$$

This completes the proof. □

Lemma 4.2. *Let $k+l>0$ and $\delta>0$. The following estimates hold*

$$\Big\|\frac{\widehat{E}^{2\delta}\big(\delta_{\mp}\widehat{\overline{A}}_{\pm,0}\big)*\big(\delta_{\pm}\widehat{A}_{\pm,0}\big)}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \leq C\|f_{\pm,0}\|_{H^\alpha}\|f_{\pm,0}\|_{H^\alpha}, \tag{4.22a}$$

$$\begin{aligned}&\Big\|\frac{\widehat{E}^{2\delta}\widehat{b}_T*\big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big)*\big(\delta_{\pm}^{(l)}\widehat{A}_{\pm,l}\big)}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \\ &\qquad\leq C(k+l)T^{k+l-\frac{1}{2}}\|f_{\pm,k}\|_{H^\alpha}\|f_{\pm,l}\|_{H^\alpha}.\end{aligned} \tag{4.22b}$$

Proof. We only demonstrate the case (4.22b) since the proof for (4.22a) is analogous. Using duality, we demonstrate the case $(+,+)$, while the case $(-,-)$ is being similar. We first compute the fractional term

$$\widehat{D}_+(\xi)\gamma^0\widehat{D}_+(\eta) = \gamma^0(|\xi||\eta|+\xi\cdot\eta)-\gamma^0\gamma^j\gamma^k\xi_j\eta_k\Big|_{j\neq k}-\gamma^j(|\eta|\xi_j+|\xi|\eta_j). \tag{4.23}$$

Thus, in the same manner we have

$$\begin{aligned}&\Big|\big\langle b_T\overline{Z}_{+,k}Z_{+,l},g\big\rangle\Big| \\ =&\Big|\int\widehat{f}^{\dagger}_{+,k}(-\xi)\frac{\widehat{D}(|\xi|,\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}b_T}g}(-|\xi|+|\eta|,\xi+\eta)\,d\xi d\eta\Big| \\ \leq&\|f_{+,k}\|_{H^\alpha}\|f_{+,l}\|_{H^\alpha}\cdot \\ &\Big(\int\frac{|\xi||\eta|+\xi\cdot\eta}{|\xi|^{1+2\alpha}|\eta|^{1+2\alpha}}\big|\widehat{t^{k+l}b_T}g(-|\xi|+|\eta|,\xi+\eta)\big|^2\,d\xi d\eta\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.24}$$

To estimate the above integral, we change the variables by

$$|\xi| - |\eta| = \tau, \quad \xi + \eta = z, \quad \xi = \rho\omega, \quad |\omega| = 1, \tag{2.25}$$

thus we can rewrite it as

$$\int H(\tau, z)\big|\widehat{t^{k+l}b_T g}(-\tau, z)\big|^2 \, d\tau dz, \tag{2.26}$$

where

$$H(\tau, z) = \int \frac{|\xi||\eta| + \xi \cdot \eta}{|\xi|^{1+2\alpha}|\eta|^{1+2\alpha}} \rho^2 \frac{d\rho}{d\tau} d\omega. \tag{2.27}$$

Throughout some computations, we have

$$H(\tau, z) \leq C\frac{|z| - \tau}{|z|^{4\delta}}, \tag{4.28}$$

$$\big\|(|\tau| + 1)^{\frac{1}{2}} \widehat{t^{k+l}b_T}\big\|_{L^1} \leq C(k + l)T^{k+l-\frac{1}{2}} \|b\|_{H^1}. \tag{4.29}$$

With the aid of the above inequalities and the observation

$$\frac{\big||\tau| - |\xi|\big| + 1}{(|\tau| + |\xi| + 1)^{4\delta}} \leq C\frac{\big||\tau - \sigma| - |\xi|\big| + 1}{(|\tau - \sigma| + |\xi| + 1)^{4\delta}} \big(|\sigma| + 1\big)^{1-4\delta}, \tag{4.30}$$

we can estimate

$$\begin{aligned} \|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}} \widehat{t^{k+l}b_T g}\|_{L^2} \leq & \big\|(|\tau| + 1)^{\frac{1}{2}} \widehat{t^{k+l}b_T}\big\|_{L^1} \Big\|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}}\widehat{g}\Big\|_{L^2} \\ \leq & C(k + l)T^{k+l-\frac{1}{2}} \Big\|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^{2\delta}}\widehat{g}\Big\|_{L^2}. \end{aligned} \tag{4.31}$$

This completes the proof. □

Remark. *Invoke the dyadic decomposition, in fact, we can show that the estimates (4.22) hold for* $\delta = 0$.

Lemma 4.3. *Let* $\delta_1 > 0$. *The following estimates hold*

$$\|f_{\pm,0}\|_{H^\alpha} \leq C\Big(\|\psi_0\|_{H^\alpha} + \Big\|\frac{\widehat{M}^\alpha \widehat{G}}{\widehat{S}^{\frac{1}{2}-\delta_1}}\Big\|_{L^2}\Big), \tag{4.32a}$$

$$\|f_{\pm,k}\|_{H^\alpha} \leq C\frac{2^k}{k!}\Big(\|\widehat{M}^\alpha \widehat{G}_+\|_{L^2} + \|\widehat{M}^\alpha \widehat{G}_-\|_{L^2}\Big). \tag{4.32b}$$

The proof for the Lemma 4.3 is straight forward so that we skip it. Notice that, in the (4.32b), $\widehat{S} \sim 1$ on the support of $\widehat{G}_\pm$.

Lemma 4.4. *With the notation above, the following estimate holds*

$$\Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_1 * \widehat{K}_1}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \leq C\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_f}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}^2. \tag{4.33}$$

Proof. For simplicity, we write $\widehat{G} := \widehat{G}_f$ and $\widehat{K} := \widehat{K}_1$. We use dyadic decomposition to handle this case. Assume that

$$\widehat{G} = \sum_{k=0}^{\infty} \widehat{G}_{\pm,\pm,k}, \tag{4.34}$$

where $\widehat{G}_{\pm,\pm,k}(\tau,\xi)$ is supported in one of the following types of regions:

$$\Sigma_{+,+} := \{(\tau,\xi) : \tau > 0, +2^{k-1} < \tau - |\xi| < +2^{k+1}\}, \tag{4.35a}$$

$$\Sigma_{+,-} := \{(\tau,\xi) : \tau > 0, -2^{k+1} < \tau - |\xi| < -2^{k-1}\}, \tag{4.35b}$$

$$\Sigma_{-,+} := \{(\tau,\xi) : \tau < 0, +2^{k-1} < \tau + |\xi| < +2^{k+1}\}, \tag{4.35c}$$

$$\Sigma_{-,-} := \{(\tau,\xi) : \tau < 0, -2^{k+1} < \tau + |\xi| < -2^{k-1}\}. \tag{4.35d}$$

They are forward and backward cones with thickness about 2^k, also half of them are truncated. The decomposition of $\widehat{G}$ induces a decomposition for $\widehat{K}$, namely

$$\widehat{K}_{\pm,\pm,k} = \frac{\widehat{D}}{\widehat{W}}\widehat{G}_{\pm,\pm,k}. \tag{4.36}$$

Let g be an arbitrary function. we compute

$$\begin{aligned}
&\Big\langle \widehat{\overline{K}}_k * \widehat{K}_l, \widehat{g}\Big\rangle \\
={}& \int \widehat{\overline{K}}_{\pm,\pm,k} * \widehat{K}_{\pm,\pm,l}(-\tau,-\xi)\overline{\widehat{g}}(-\tau,-\xi)d\tau d\xi \\
={}& \int \widehat{\overline{K}}_{\pm,\pm,k}(-\tau-\sigma,-\xi-\eta)\widehat{K}_{\pm,\pm,l}(\sigma,\eta)\overline{\widehat{g}}(-\tau,-\xi)\, d\sigma d\eta d\tau d\xi \\
={}& \int \widehat{K}^{\dagger}_{\pm,\pm,k}(\tau+\sigma,\xi+\eta)\gamma^0\widehat{K}_{\pm,\pm,l}(\sigma,\eta)\overline{\widehat{g}}(-\tau,-\xi)\, d\sigma d\eta d\tau d\xi,
\end{aligned} \tag{4.37}$$

we have 16 cases resulted from (4.35a-d) and (4.36b). Due to the support of the integrand, the integral is integrated over one of the sets of $(\tau, \sigma, \xi, \eta)$:

$$\{\tau+\sigma>0,\, \sigma>0,\, \tau+\sigma-|\xi+\eta| \sim \pm 2^k,\, \sigma-|\eta| \sim \pm 2^l\}, \tag{4.38a}$$

$$\{\tau+\sigma<0,\, \sigma<0,\, \tau+\sigma+|\xi+\eta| \sim \pm 2^k,\, \sigma+|\eta| \sim \pm 2^l\}, \tag{4.38b}$$

$$\{\tau+\sigma<0,\, \sigma>0,\, \tau+\sigma+|\xi+\eta| \sim \pm 2^k,\, \sigma-|\eta| \sim \pm 2^l\}, \tag{4.38c}$$

$$\{\tau+\sigma>0,\, \sigma<0,\, \tau+\sigma-|\xi+\eta| \sim \pm 2^k,\, \sigma+|\eta| \sim \pm 2^l\}. \tag{4.38d}$$

We label them as

$$\Sigma_{k,l}[(\pm,\pm);(\pm,\pm)], \tag{4.39}$$

and denote by $\Sigma_{k,l}$ without specifying which one precisely. We also use $\widehat{K}_k$ for abbreviation of $\widehat{K}_{\pm,\pm,k}$ and $\widehat{G}_k$ for $\widehat{G}_{\pm,\pm,k}$.

We first compute

$$\begin{aligned} &\widehat{D}(\tau+\sigma,-\xi-\eta)\gamma^0\widehat{D}(\sigma,\eta) \\ =\ &\big[\gamma^0(\tau+\sigma)-\gamma^j(\xi_j+\eta_j)\big]\gamma^0\big[\gamma^0\sigma+\gamma^j\eta_j\big] \\ =\ &\gamma^0\big[(\tau+\sigma)\sigma-(\xi+\eta)\cdot\eta\big]+\gamma^0\gamma^j\gamma^k(\xi_j+\eta_j)\eta_k\Big|_{j\neq k} \\ &+\gamma^j\big[(\tau+\sigma)\eta_j-\sigma(\xi_j+\eta_j)\big]. \end{aligned} \tag{4.40}$$

Thus, we have

$$\begin{aligned} &\Big|\Big\langle \widehat{\overline{K}}_k * \widehat{K}_l, \widehat{g}\Big\rangle\Big| \\ =&\Big|\int \widehat{G}_k^{\dagger}(\tau+\sigma,\xi+\eta)\frac{\gamma^0(\tau+\sigma)-\gamma^j(\xi_j+\eta_j)}{(\tau+\sigma)^2-(\xi+\eta)^2}\gamma^0\frac{\gamma^0\sigma+\gamma^j\eta_j}{\sigma^2-\eta^2}\widehat{G}_l(\sigma,\eta)\cdot \\ &\overline{\widehat{g}}(-\tau,-\xi)d\sigma d\eta d\tau d\xi\Big| \\ \leq& C\|\widehat{M}^{\alpha}\widehat{G}_k\|_{L^2}\|\widehat{M}^{\alpha}\widehat{G}_l\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau d\xi\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.41}$$

where $I_{k,l}(\tau,\xi)$ is given by

$$I_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)Q(\tau,\sigma,\xi,\eta)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma d\eta, \tag{4.42}$$

and Q is given by the expression

$$\begin{aligned} &Q(\tau,\sigma,\xi,\eta) := \\ &\big[(\tau+\sigma)\sigma-(\xi+\eta)\cdot\eta\big]^2+\big|(\xi+\eta)\times\eta\big|^2+\big|(\tau+\sigma)\eta-\sigma(\xi+\eta)\big|^2, \end{aligned} \tag{4.43}$$

and $D_{k,l}(\tau,\xi)$ is a slice of $\Sigma_{k,l}$ for fixed (τ,ξ), i.e.

$$D_{k,l}(\tau,\xi) := \{(\sigma,\eta) : (\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}\}. \tag{4.44}$$

They are the intersections of cones with thickness 2^k and cones with thickness 2^l.

We distinguish the cases into two sets,

$$\Sigma_{k,l}[(\pm,\cdot);(\pm,\cdot)] \quad \text{and} \quad \Sigma_{k,l}[(\pm,\cdot);(\mp,\cdot)], \tag{4.45}$$

due to the fact that for the first set it is the intersection of two forward cones or two backward cones; for the second set it is the intersection of a forward cone and a backward cone. Therefore presumably the computations for the 8 cases in each set are similar.

Cases H. *We have the following estimate*

$$\begin{aligned} \Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{+,\cdot,k}*\widehat{K}_{+,\cdot,l}}{\widehat{S}^{\frac12}}\Big\|_{L^2} &\le \frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M^{\alpha}}\widehat{G}_{+,\cdot,k}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M^{\alpha}}\widehat{G}_{+,\cdot,l}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}, \\ \Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{-,\cdot,k}*\widehat{K}_{-,\cdot,l}}{\widehat{S}^{\frac12}}\Big\|_{L^2} &\le \frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M^{\alpha}}\widehat{G}_{-,\cdot,k}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M^{\alpha}}\widehat{G}_{-,\cdot,l}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}. \end{aligned} \tag{4.46}$$

In these cases, we have $(\tau+\sigma)\sigma>0$. Take the case

$$\widehat{\overline{K}}_{+,+,k}*\widehat{K}_{+,+,l}, \tag{4.47}$$

as an example and in which $D_{k,l}=\{(\sigma,\eta):\tau+\sigma>0,\sigma>0,\tau+\sigma-|\xi+\eta|\sim 2^k,\sigma-|\eta|\sim 2^l,(\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}[(+,+);(+,+)]\}$. In the $\eta\sigma$-space, this is the region of the intersection of two forward cones, which is unbounded.

To manage the integral, we change the variables:

$$h = |\xi+\eta| - |\eta|, \qquad 2X = h\cosh\zeta, \tag{4.48a}$$
$$2Y = \sqrt{|\xi|^2-h^2}\sinh\zeta\cos\varphi,\ 2Z = \sqrt{|\xi|^2-h^2}\sinh\zeta\sin\varphi. \tag{4.48b}$$

This is due to the fact that for fixed ξ and h, η stays on an hyperboloid. The coordinates (X, Y, Z) identity the vector η as follows $\eta = (X, Y, Z) - (|\xi|/2, 0, 0)$ and $\xi + \eta = (X, Y, Z) + (|\xi|/2, 0, 0)$. Hence $d\eta = dXdYdZ = Jdhd\alpha d\varphi$, where J is the Jacobian given by

$$8J = 8\frac{\partial(X,Y,Z)}{\partial(h,\zeta,\varphi)} = \big(|\xi|^2\cosh^2\zeta - h^2\big)|\sinh\zeta| = 4|\xi+\eta||\eta||\sinh\zeta|. \tag{4.49}$$

Throughout some algebraic manipulations, the Q can be rewritten as

$$\begin{aligned}2Q =&(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2 + (\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2+\\ &4(\tau+\sigma)\sigma(|\xi|^2-h^2).\end{aligned} \tag{4.50}$$

Now the integral $I_{k,l}$ can be split into three parts according to (4.50). To estimate each term separately, we consider a further dyadic decomposition

$$\begin{aligned}&\widehat{G}_{k,m}(\tau,\xi) \quad \text{is supported in} \quad \{\tau>0,\ \tau-|\xi|\sim 2^k,\ |\xi|\sim 2^m\},\\ &\widehat{G}_{l,n}(\tau,\xi) \quad \text{is supported in} \quad \{\tau>0,\ \tau-|\xi|\sim 2^l,\ |\xi|\sim 2^n\}.\end{aligned}$$

Therefore the integral in (4.29b) is over the region $D^{m,n}_{k,l}(\tau,\xi)$ which is the set of (σ,η):

$$\{\tau+\sigma>0, \tau>0, \tau+\sigma-|\xi+\eta|\sim 2^k, \sigma-|\eta|\sim 2^l, |\xi+\eta|\sim 2^m, |\eta|\sim 2^n\}. \tag{4.51}$$

For simplicity, we will assume $k \geq l$, $m \geq n$, and $n > k$ while the other case is similar.

Claim. *The size of the set $D^{m,n}_{k,l}$ is bounded as follows*

$$\big|D^{m,n}_{k,l}\big| = \int_{D^{m,n}_{k,l}} d\sigma d\eta \leq C2^{2n}2^{k+l}. \tag{4.52}$$

Also $\widehat{E}(\tau,\xi) \geq 2^k$ *for* $k \gg l$ *and*

$$\int_{D_{k,l}^{m,n}} |\xi|^2 - h^2 \, d\sigma d\eta \leq C 2^{2k+l} 2^{m+2n} \widehat{S}. \tag{4.53}$$

The proof of the claim will be given in the appendix. Assuming the claim, we can estimate the first part:

$$\begin{aligned}
&I^1_{k,l,m,n}(\tau,\xi) \\
:=&\int_{D_{k,l}^{m,n}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
=&\int_{D_{k,l}^{m,n}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma d\eta \\
\leq&\frac{C}{2^{2l}} \int_{D_{k,l}^{m,n}} \frac{(|\xi+\eta|+1)^{-2\alpha}(|\eta|+1)^{-2\alpha}}{(2^k+|\xi+\eta|)^2} d\sigma d\eta \\
\leq&\frac{C}{2^{2l}} \frac{2^{-2\alpha n}}{2^{(2+2\alpha)m}} \int_{D_{k,l}^{m,n}} d\sigma d\eta \leq \frac{C}{2^{2l}} \frac{2^{-2\alpha n}}{2^{(2+2\alpha)m}} 2^{2n} 2^{k+l} \\
\leq&\frac{C}{2^{-k+l}} \frac{2^{(2-2\alpha)n}}{2^{(2+2\alpha)m}} \frac{\widehat{S} 2^{2\delta m}}{\widehat{E}^{2\delta}} \leq \frac{C}{2^{-k+l}} \frac{1}{2^{(2-2\alpha)(m-n)} 2^{4\alpha m - 2\delta m}} \frac{\widehat{S}}{\widehat{E}^{2\delta}} \\
\leq&\frac{C}{2^{2\delta m + l}} \frac{1}{2^{(2-2\alpha)(m-n)} 2^{m-k}} \frac{\widehat{S}}{\widehat{E}^{2\delta}}.
\end{aligned} \tag{4.54a}$$

For the second part, we obtain

$$\begin{aligned}
&I^2_{k,l,m,n}(\tau,\xi) \\
:=&\int_{D_{k,l}^{m,n}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
=&\int_{D_{k,l}^{m,n}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
\leq&\frac{C}{2^{2k}} \frac{2^{-2\alpha m}}{2^{(2+2\alpha)n}} \int_{D_{k,l}^{m,n}} d\sigma d\eta \leq \frac{C}{2^{2k}} \frac{2^{-2\alpha m}}{2^{(2+2\alpha)n}} 2^{2n} 2^{k+l} \\
\leq&\frac{C}{2^{k-l}} \frac{1}{2^{2\alpha(m+n)}} \frac{\widehat{S} 2^{2\delta m}}{\widehat{E}^{2\delta}} \leq \frac{C}{2^{k-l}} \frac{1}{2^{\frac{1}{2}m + 2\alpha n}} \frac{\widehat{S}}{\widehat{E}^{2\delta}}.
\end{aligned} \tag{4.54b}$$

For the third part, we get

$$
\begin{aligned}
&I^3_{k,l,m,n}(\tau,\xi)\\
:=&\int_{D^{m,n}_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma)\sigma\big(|\xi|^2-h^2\big)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta\\
\leq&\frac{C}{2^{2k+2l}}\int_{D^{m,n}_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma)\sigma\big(|\xi|^2-h^2\big)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta\\
\leq&\frac{C}{2^{2k+2l}}\int_{D^{m,n}_{k,l}} \frac{|\xi|^2-h^2}{(|\xi+\eta|+1)^{1+2\alpha}(|\eta|+1)^{1+2\alpha}}\ d\sigma d\eta\\
\leq&\frac{C}{2^{2k+2l}}\,\frac{1}{2^{(1+2\alpha)(m+n)}}\int_{D^{m,n}_{k,l}} |\xi|^2-h^2\ d\sigma d\eta\\
\leq&\frac{C}{2^{2k+2l}}\ \frac{1}{2^{(1+2\alpha)(m+n)}}\ 2^{2k+l}\,2^{m+2n}\widehat{S}\ \frac{2^{2\delta m}}{\widehat{E}^{2\delta}}\\
\leq&\frac{C}{2^l}\,\frac{1}{2^{\frac{1}{2}(m-n)+2\delta n}}\,\frac{\widehat{S}}{\widehat{E}^{2\delta}}.
\end{aligned}
\tag{4.54c}
$$

Combine the above results , we have

$$
\begin{aligned}
&\Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{+,+,k} * \widehat{K}_{+,+,l}}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \leq \sum_{m,n}\Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{+,+,k,m} * \widehat{K}_{+,+,l,n}}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2}\\
\leq&\sum_{m,n}\frac{1}{2^{\frac{1}{4}|m-n|}}\,\frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,+,k,m}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,+,l,n}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\\
\leq&\frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,+,k}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,+,l}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}.
\end{aligned}
\tag{4.55}
$$

Cases E. *We have the following estimate*

$$
\begin{aligned}
&\Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{-,\cdot,k} * \widehat{K}_{+,\cdot,l}}{\widehat{S}^{\frac{1}{2}}}\Big\|_{L^2} \leq \frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{-,\cdot,k}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,\cdot,l}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2},\\
&\Big\|\frac{\widehat{E}^{\delta}\widehat{\overline{K}}_{+,\cdot,k} * \widehat{K}_{-,\cdot,l}}{\widehat{S}^{\alpha}}\Big\|_{L^2} \leq \frac{C}{2^{\frac{\delta}{2}(k+l)}}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{+,\cdot,k}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^{\alpha}\widehat{G}_{-,\cdot,l}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}.
\end{aligned}
\tag{4.56}
$$

In these cases, we have $(\tau+\sigma)\sigma < 0$. We will only demonstrate the case of

$$
\widehat{\overline{K}}_{-,+,k} * \widehat{K}_{+,+,l},
\tag{4.57}
$$

and in which $D_{k,l} = \{(\sigma,\eta) : \tau+\sigma+|\xi+\eta| \sim 2^k, \sigma-|\eta| \sim 2^l, (\tau,\sigma,\xi,\eta) \in \Sigma_{k,l}[(-,+);(+,+)]\}$. In this case $\tau+\sigma<0$ and $\sigma>0$. In $\eta\sigma$-space, this is the region of the intersection of a forward cone with a truncated backward cone, which is bounded.

To manage the integral, we change the variables:

$$e = |\xi+\eta| + |\eta|, \qquad 2X = e\cos\theta, \tag{4.58a}$$

$$2Y = \sqrt{e^2-|\xi|^2}\sin\theta\cos\varphi, \quad 2Z = \sqrt{e^2-|\xi|^2}\sin\theta\sin\varphi. \tag{4.58b}$$

This is due to the fact that for fixed ξ and h, η stays on an ellipsoid. The coordinates (X,Y,Z) identity the vector η as follows $\eta = (X,Y,Z) - (|\xi|/2,0,0)$ and $\xi+\eta = (X,Y,Z) + (|\xi|/2,0,0)$. Hence $d\eta = dXdYdZ = Jded\theta d\varphi$, where J is the Jacobian given by

$$8J = 8\frac{\partial(X,Y,Z)}{\partial(e,\theta,\varphi)} = \big(e^2-|\xi|^2\cos^2\theta\big)|\sin\theta| = 4|\xi+\eta||\eta||\sin\theta|. \tag{4.59}$$

Throughout some algebraic manipulations, the Q can be rewritten as

$$\begin{aligned} 2Q =&(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2 + (\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2 + \\ &4(\tau+\sigma)\sigma(e^2-|\xi|^2). \end{aligned} \tag{4.60}$$

Now the integral $I_{k,l}$ can be split into three parts according to (4.60). For simplicity, we will assume $k \geq l$, while the other case is similar.

Claim.

$$\int d\sigma \sim 2^l, \quad \sigma \geq 2^l, \quad |\xi+\eta| \geq 2^k, \quad \widehat{E}(\tau,\xi) \geq 2^k, \tag{4.61}$$

$$\int_0^\pi \frac{(e-|\xi|\cos\theta)^{1-2\alpha}\sin\theta}{(e+|\xi|\cos\theta)^{1+2\alpha}}d\theta \leq \frac{C}{(e^2-|\xi|^2)^{2\alpha}}, \tag{4.62}$$

$$\int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}} \frac{1}{(e^2-|\xi|^2)^{2\alpha}}de \leq \frac{C}{2^{2\delta k}\widehat{E}^{2\delta}}, \tag{4.63}$$

$$\int_0^\pi \frac{(e+|\xi|\cos\theta)^{1-2\alpha}}{(e-|\xi|\cos\theta)^{2\alpha}}\sin\theta\, d\theta \leq \frac{C}{e^{4\alpha-1}}. \tag{4.64}$$

The proof of the claim will be given in the appendix.

Now for the first part, we can estimate

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
\le& \frac{1}{2^{2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{-2\alpha}(|\eta|+1)^{-2\alpha}}{(\tau+\sigma-|\xi+\eta|)^2} d\sigma d\eta \\
\le& \frac{1}{2^{2l}} \int_{D_{k,l}} \frac{1}{|\xi+\eta|^{2+2\alpha}(|\eta|+1)^{2\alpha}} d\sigma d\eta \\
\le& \frac{C}{2^{l}} \int \frac{|\xi+\eta||\eta||\sin\theta|}{|\xi+\eta|^{2+2\alpha}(|\eta|+1)^{2\alpha}} d\varphi d\theta de \\
\le& \frac{C}{2^{l}} \int \frac{(e-|\xi|\cos\theta)^{1-2\alpha}|\sin\theta|}{(e+|\xi|\cos\theta)^{1+2\alpha}} d\theta de \\
\le& \frac{C}{2^{l}} \int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}} \frac{1}{(e^2-|\xi|^2)^{2\alpha}} de \le \frac{C}{2^l 2^{2\delta k}\widehat{E}^{2\delta}}.
\end{aligned} \tag{4.65a}$$

The argument for the possible cases of $-\tau+2^{k-1}-2^{l+1}<|\xi|$, is not harder than that in (4.65a).

For the second part, we derive

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
\le& \frac{C}{2^{2k}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{-2\alpha}(|\eta|+1)^{-2\alpha}}{(\sigma+|\eta|)^2} d\sigma d\eta \\
\le& \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{1}{(2^l+|\eta|)^2(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}} d\eta \\
\le& \frac{C}{2^{2k}} \int \frac{|\xi+\eta||\eta||\sin\theta|}{(2^l+|\eta|)(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}} d\varphi d\theta de \\
\le& \frac{C}{2^{2k}} \int \frac{|\xi+\eta|^{1-2\alpha}}{(|\eta|)^{2\alpha}} |\sin\theta|\, d\theta de \\
\le& \frac{C}{2^{2k}} \int \frac{(e+|\xi|\cos\theta)^{1-2\alpha}}{(e-|\xi|\cos\theta)^{2\alpha}} |\sin\theta|\, d\theta de \\
\le& \frac{C}{2^{2k}} \int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}} \frac{1}{e^{4\alpha-1}}\, de \le \frac{C}{2^k}\frac{1}{\widehat{E}^{4\delta}}.
\end{aligned} \tag{4.65b}$$

Again the argument for the possible cases of $-\tau + 2^{k-1} - 2^{l+1} < |\xi|$, is not harder than that in (4.65b).

For the third part, we have

$$\begin{aligned}
I^3_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)|\tau+\sigma|\sigma(e^2-|\xi|^2)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
\leq& \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(e^2-|\xi|^2)}{|\tau+\sigma-|\xi+\eta||(\sigma+|\eta|)} \, d\sigma d\eta \\
\leq& \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{\widehat{M}^{-2\alpha}(\xi+\eta)\widehat{M}^{-2\alpha}(\eta)(e^2-|\xi|^2)}{|\xi+\eta|(2^l+|\eta|)} \, d\eta \\
\leq& \frac{C}{2^{2k+l}} \int \frac{(e^2-|\xi|^2)|\sin\theta|}{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}} \, d\theta de \\
\leq& \frac{C}{2^{2k+l}} \int \frac{(e^2-|\xi|^2)|\sin\theta|}{(e^2-|\xi|^2\cos^2\theta)^{2\alpha}} \, d\theta de \\
\leq& \frac{C}{2^{2k+l}} \int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}} \frac{e^2-|\xi|^2}{e^{4\alpha}} \, de \\
\leq& \frac{C}{2^{2k+l}} \int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}} \frac{e-|\xi|}{e^{4\alpha-1}} \, de \leq \frac{C}{2^l} \frac{1}{2^{2\delta k}} \frac{\widehat{S}}{\widehat{E}^{2\delta}}.
\end{aligned} \tag{4.65c}$$

Now we return to the proof of (4.33). Combine (4.46), (4.56), we get

$$\begin{aligned}
\Big|\langle \overline{K}_k K_l, g\rangle\Big| \leq& C\Big\|\widehat{M}^\alpha \widehat{G}_k\Big\|_{L^2}\Big\|\widehat{M}^\alpha \widehat{G}_l\Big\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau d\xi\Big)^{\frac{1}{2}} \\
\leq& \frac{C}{2^{\frac{1}{2}l}} \frac{1}{2^{\delta k}} \Big\|\widehat{M}^\alpha \widehat{G}_k\Big\|_{L^2}\Big\|\widehat{M}^\alpha \widehat{G}_l\Big\|_{L^2} \|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^\delta}\widehat{g}\|_{L^2} \\
\leq& \frac{C}{2^{(\frac{1}{2}-\frac{\delta}{2})l+\frac{\delta}{2}k}} \Big\|\frac{\widehat{M}^\alpha \widehat{G}_k}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^\alpha \widehat{G}_l}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2} \|\frac{\widehat{S}^{\frac{1}{2}}}{\widehat{E}^\delta}\widehat{g}\|_{L^2}.
\end{aligned} \tag{4.66}$$

Finally, we have

$$\begin{aligned}
&\Big\|\frac{\widehat{E}^\delta}{\widehat{S}^{\frac{1}{2}}}\widehat{\overline{K}} * \widehat{K}\Big\|_{L^2} \leq \sum_{k,l} \Big\|\frac{\widehat{E}^\delta}{\widehat{S}^{\frac{1}{2}}}\widehat{\overline{K}}_k * \widehat{K}_l\Big\|_{L^2} \\
\leq& \sum_{k,l} \frac{C}{2^{(\frac{1}{2}-\frac{\delta}{2})l+\frac{\delta}{2}k}} \Big\|\frac{\widehat{M}^\alpha \widehat{G}_k}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2}\Big\|\frac{\widehat{M}^\alpha \widehat{G}_l}{\widehat{S}^{\frac{\delta}{2}}}\Big\|_{L^2} \leq C\Big\|\frac{\widehat{M}^\alpha \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}}\Big\|^2_{L^2}.
\end{aligned} \tag{4.67}$$

This completes the proof. □

The estimates for the remaining cases are given in the following Lemma.

Lemma 4.5. *For $j = 1, 2$ and $k = 0, 1, 2, \cdots$. The following estimates hold*

$$\Big\| \frac{\widehat{E}^{\delta}}{\widehat{S}^{\frac{1}{2}}} \widehat{b}_T * \big(\delta_{\mp}^{(k)} \widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_j\big) \Big\|_{L^2} \le C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^\alpha} \Big\| \frac{\widehat{M^\alpha} \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2}, \tag{4.68}$$

$$\Big\| \frac{\widehat{E}^{\delta}}{\widehat{S}^{\frac{1}{2}}} \widehat{b}_T \widehat{\overline{K}}_j * \big(\delta_{\pm}^{(k)} \widehat{A}_{\pm,k}\big) \Big\|_{L^2} \le C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^\alpha} \Big\| \frac{\widehat{M^\alpha} \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2}, \tag{4.69}$$

$$\Big\| \frac{\widehat{E}^{\delta}}{\widehat{S}^{\frac{1}{2}}} \widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_2 \Big\|_{L^2} \le C \Big\| \frac{\widehat{M^\alpha} \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2}^2, \tag{4.70}$$

$$\Big\| \frac{\widehat{E}^{\delta}}{\widehat{S}^{\frac{1}{2}}} \widehat{b}_T * \widehat{\overline{K}}_2 * \widehat{K}_j \Big\|_{L^2} \le C \Big\| \frac{\widehat{M^\alpha} \widehat{G}}{\widehat{S}^{\frac{\delta}{2}}} \Big\|_{L^2}^2, \tag{4.71}$$

The proof of Lemma 4.5 is a repetition of the arguments presented in Lemmas 4.1, 4.2, and 4.4, so that we omit it.

Proof of Claim.

Notice that since $h = |\xi + \eta| - |\eta| = -(\tau + \sigma - |\xi + \eta|) + (\sigma - |\eta|) + \tau$, thus we have

Proof of (4.61):

$$2^{l-1} + |\eta| < \sigma < 2^{l+1} + |\eta| \quad \text{implies} \quad \int d\sigma \sim 2^l. \tag{4.72}$$

Proof of (4.52):

$$\begin{aligned} |D_{k,l}^{m,n}| \le & \int_{D_{k,l}^{m,n}} d\sigma d\eta \le 2^l \int_{\widetilde{D}_{k,l}^{m,n}} d\eta \le 2^l \int |\xi + \eta| |\eta| |\sinh \zeta| \, d\varphi d\zeta dh \\ \le & 2^l \, 2^{m+n} \int \int_{\zeta_1}^{\zeta_2} |\sinh \zeta| \, d\zeta dh \\ \le & 2^l \, 2^{m+n} \int \cosh \zeta_2 - \cosh \zeta_1 \, dh \\ \le & 2^l \, 2^{m+n} \int_{\tau - 2^{k+1} + 2^{l-1}}^{\tau - 2^{k-1} + 2^{l+1}} \frac{h + 2|\eta_2|}{|\xi|} - \frac{h + 2|\eta_1|}{|\xi|} \, dh \\ \le & 2^l \, 2^{m+n} \frac{2^n}{2^m} \int_{\tau - 2^{k+1} + 2^{l-1}}^{\tau - 2^{k-1} + 2^{l+1}} dh \le C 2^{k+l} \, 2^{2n}. \end{aligned} \tag{4.73}$$

The proof for (4.53) is analogous with the one given above.

Proof of (4.62): Let $\lambda = |\xi|/e$. Change of variable $t = 1 + \lambda\cos\theta$ gives

$$\begin{aligned}\int_0^\pi \frac{(e-|\xi|\cos\theta)^{1-2\alpha}}{(e+|\xi|\cos\theta)^{1+2\alpha}}\sin\theta\,d\theta \leq & \frac{1}{e^{4\alpha}}\int_{1-\lambda}^{1+\lambda}\frac{(2-t)^{1-2\alpha}}{t^{1+2\alpha}}\frac{1}{\lambda}dt \\ \leq & \frac{C}{(e^2-|\xi|^2)^{2\alpha}}.\end{aligned} \tag{4.74}$$

Proof of (4.63):

$$\begin{aligned}\int_{-\tau+2^{k-1}-2^{l+1}}^{-\tau+2^{k+1}-2^{l-1}}\frac{1}{(e^2-|\xi|^2)^{2\alpha}}de \leq & \frac{2^k}{\left((-\tau+2^{k-1}-2^{l+1})^2-|\xi|^2\right)^{2\alpha}} \\ \leq & \frac{C}{2^{2\delta k}\widehat{E}^{2\delta}}\end{aligned} \tag{4.75}$$

The proof for (4.64) can be proved in the same manner. □

Before proving the (2.36), we first state an estimate for an oscillatory integral and whose proof can be found in Stein's and Sogge's books, see [S] and [So].

Lemma 4.6. *Assume that the critical points set $\nabla\phi = 0$ is such that the Hessian $\nabla^2\phi$ has rank $n-1$. Then*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\phi(\xi)}b(\xi)\,d\xi\right| \leq \frac{C(\phi,b)}{|\lambda|^{(n-1)/2}}. \tag{4.76}$$

Proof of (2.36):

Equip with (4.76) and (2.34a), for $k = 0,1,2,\cdots$ we obtain

$$\begin{aligned}&\left|\mathcal{F}^{-1}\Big(\widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\widehat{\varphi}_k\widehat{g}_k\Big)(t,x)\right| \\ =&\Big|\int e^{ix\xi}\int\mathcal{F}_\xi^{-1}\Big(\widehat{E}^{-2\gamma}\widehat{S}^{-2\beta}\Big)(t-s)\widetilde{g}_k(s,\xi)\,ds\widehat{\varphi}_k(\xi)\,d\xi\Big| \\ \sim&\Big|\int\int e^{ix\xi}\frac{e^{\pm i(t-s)|\xi|}}{(|\xi|+1)^{2\gamma}}\widehat{\varphi}_k(\xi)\widetilde{g}_k(s,\xi)\,d\xi\frac{\ell(t-s)}{|t-s|^{1-2\beta}}\,ds\Big| \\ \leq&\int\int\Big|\int e^{i(x-y)\xi\pm i(t-s)|\xi|}\frac{\widehat{\varphi}_k(\xi)}{(|\xi|+1)^{2\gamma}}\,d\xi\Big||g_k(s,y)|\,dy\frac{|\ell(t-s)|}{|t-s|^{1-2\beta}}\,ds \\ \leq&\int\int\frac{C(\widehat{\varphi})2^{2(1-\gamma)k}}{|t-s|}|g_k(s,y)|\,dy\frac{C}{|t-s|^{1-2\beta}}\,ds.\end{aligned} \tag{4.77}$$

This completes the proof of the estimate (2.36). ☐

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