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On a Bilinear Estimate of Schrödinger Waves

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Abstract

In this paper we want to consider a bilinear space-time estimate for homogeneous Schrödinger equations. We give an elementary proof for the estimates in Bourgain space, which is in a form of scaling invariance.

1 Introduction

Consider the homogeneous Schrödinger equations

$$\begin{cases} iu_t - \Delta u = 0, \ (x,t) \in R^n \times R, \\ u(0) = f; \end{cases} \qquad \begin{cases} iv_t - \Delta v = 0, \ (x,t) \in R^n \times R, \\ v(0) = g. \end{cases} \tag{1.1}$$

Via Fourier transform, the solution u and v can be written as

$$u(t) = e^{-it\Delta} f \quad \text{and} \quad v(t) = e^{-it\Delta} g. \tag{1.2}$$

Thus to study the estimates of the product of Schrödinger waves, uv, is to study the estimates of the product

$$e^{-it\Delta} f e^{-it\Delta} g. \tag{1.3}$$

There are many literature investigating on the topic of bilinear estimates for Schödinger waves. In '98, Ozawa and Tsutsumi [OT] proved an L^2 estimate for $u\bar{v}$ with $\frac{1}{2}$ derivative for $n = 1$,

$$\left\| (-\Delta)^{\frac{1}{4}} (e^{it\Delta} f)(e^{-it\Delta} g) \right\|_{L^2_{t,x}} = \frac{1}{\sqrt{2}} \|f\|_{L^2} \|g\|_{L^2}. \tag{1.4}$$

In '98, Bourgain [Bo] showed a refinements of Strichartz' inequality for $n = 2$. If $\widehat{f}$ is supported on $|\xi| \sim N$, $\widehat{g}$ is supported on $|\xi| \sim M$, and $M << N$, then

$$\|(e^{it\Delta} f)(e^{\pm it\Delta} g)\|_{L^2_{t,x}} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}. \tag{1.5}$$

In '01, Kenig etc. [CDKS] obtained a bilinear estimate in Bourgain space for nonlinear Schrödinger equation in two dimension. Let $b = \frac{1}{2}+$. If $-\frac{1}{4} - (1-b) < s$ and $\sigma < \min\big(s + \frac{1}{2}, 2s + 2(1-b)\big)$, then

$$\|uv\|_{X^{\sigma,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}. \tag{1.6}$$

In '03, Tao [T1] obtained a sharp bilinear restriction estimate for paraboloids. Let $q > \dfrac{n+3}{n+1}$, $n \geq 2$, $N > 0$, and f and g have Fourier transform supported in the region $|\xi| \leq N$. Suppose that $dist(supp\,\widehat{f}, supp\,\widehat{g}) \geq cN$. Then we have

$$\|e^{-it\Delta}fe^{-it\Delta}g\|_{L^q_{t,x}(R^{n+1})} \lesssim N^{n-\frac{n+2}{q}}\|f\|_{L^2}\|g\|_{L^2}. \tag{1.7}$$

In '05, Burq, Jérard, and Tzvetkov [BGT] derived bilinear eigenfunction estimates on sphere and on Zoll surfaces. In '09, Keraani and Vargas [KV] showed a bilinear estimate of uv in $L^{\frac{n+2}{n}}$ norm, where $n \geq 2$. If $b \in (0, \frac{2}{n+2})$, then

$$\|e^{-it\Delta}fe^{-it\Delta}g\|_{L^{\frac{n+2}{n}}(R^{n+1})} \lesssim C\|f\|_{\dot{H}^b}\|g\|_{\dot{H}^{1-b}}. \tag{1.8}$$

In '09, Kishimoto [K] derived an improved bilinear estimate for quadratic Schrödinger equation in one and two dimensions. The estimate is in a variant of Bourgain space with weighted norm. In '10, Chae, Cho, and Lee [CCL] proved an interactive estimate of uv in a mixed norm. Let $n \geq 2$. If $\frac{2}{q} = n(1-\frac{1}{r})$, $1 < r \leq 2$, $\quad q > 1$, $|s| < 1 - \frac{1}{r}$, then

$$\|e^{-it\Delta}fe^{-it\Delta}g\|_{L^q_tL^r_x} \leq C\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^{-s}}. \tag{1.9}$$

Let D, S_+, and S_- be the operators with the symbols

$$\widehat{D} \stackrel{\text{def}}{=} |\xi|, \quad \widehat{S}_+ \stackrel{\text{def}}{=} \big||\tau| + |\xi|^2\big|, \quad \text{and} \quad \widehat{S}_- \stackrel{\text{def}}{=} \big||\tau| - |\xi|^2\big|, \tag{1.10}$$

respectively.

Theorem 1. *Let $n \geq 2$. If for $j = 1, 2$,*

$$\begin{gathered} \beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2, \\ \beta_- \geq 0, \quad \beta_- - \alpha_j + \frac{n-1}{2} \geq 0, \\ (\beta_-, \alpha_j) \neq (0, \frac{n-1}{2}), \ \textit{and} \quad \beta_0 > -\frac{n-1}{2}, \end{gathered} \tag{1.11}$$

then

$$\left\|D^{\beta_0}S_+^{\beta_+}S_-^{\beta_-}\left(e^{-it\Delta}fe^{-it\Delta}g\right)\right\|_{L^2(R^{n+1})} \leq C\|f\|_{\dot{H}^{\alpha_1}(R^n)}\|g\|_{\dot{H}^{\alpha_2}(R^n)}. \tag{1.12}$$

Notice that Strichartz Estimate for Homogeneous Schrödinger equation for $n = 2$ reads

$$\|u\|_{L^4} \lesssim \|f\|_{L^2},$$

which coincides with the bilinear estimate

$$\|u\|^2_{L^4} = \|uu\|_{L^2} \lesssim \|f\|_{L^2}\|f\|_{L^2},$$

when

$$\beta_0 = \beta_+ = \beta_- = \alpha_1 = \alpha_2 = 0.$$

The estimate is given in the form of scaling invariance. The conditions stated in the theorem come from the scaling invariance and interactions between frequencies.

The proof of Theorem 1 is based on the ideas of the work of Foschi and Klainerman [FK], and the work of Klainerman and Machedon [KM], however some modifications for adapting the case of Schrödinger are required. The purpose of this work is to derive a new estimate with an elementary proof.

The paper is organized as follows: In Section 2, we prove Theorem 1. In Section 3, we state and prove some properties which are the technical parts left in the proof of Theorem 1.

2 Bilinear Estimates for Schrödinger waves

We denote the Fourier transform of the function $u(t,x)$ by $\widehat{u}(t,\xi)$ with respect to the space variable and by $\widetilde{u}(\tau,\xi)$ with respect to the space-time variables. For simplicity, we call

$$\widehat{A} \equiv \widehat{D}^{2\beta_0}\,\widehat{S}_+^{2\beta_+}\,\widehat{S}_-^{2\beta_-}.$$

We now prove Theorem 1.

Proof. First we compute the Fourier transform of the product $e^{-it\Delta}fe^{-it\Delta}g$ with respect to space variables,

$$\widehat{e^{-it\Delta}f} * \widehat{e^{-it\Delta}g}(\xi) = \int e^{it|\xi-\eta|^2}\widehat{f}(\xi-\eta)e^{it|\eta|^2}\widehat{g}(\eta)d\eta = \int e^{it(|\xi-\eta|^2+|\eta|^2)}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta.$$

Thus its Fourier transform with respect to space-time variables is

$$\int \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta,$$

where $\delta(\tau - |\xi-\eta|^2 - |\eta|^2)d\eta$ is viewed as a measure supported on surfaces $\{\eta : \tau = |\xi-\eta|^2+|\eta|^2\}$.

We split the integral into three parts in the following way. First we define a function

$$h(\gamma) \overset{\text{def}}{=} \frac{\sqrt{2\gamma-1}}{\gamma} \tag{2.1}$$

which will appear in the proof later, see figure 1. Since the equation $h(\gamma) = 1/3$ has two roots $9 \pm 6\sqrt{2}$, we denote the two roots by $\gamma_1 = 9-6\sqrt{2}$ and $\gamma_2 = 9+6\sqrt{2}$. Then we decompose the η-space into $S_a \cup S_b \cup S_c$, see figure 2, where

$$\begin{aligned}
&S_a \overset{\text{def}}{=} \{\eta : \tfrac{1}{2}|\xi|^2 \le |\xi-\eta|^2+|\eta|^2 \le \gamma_1|\xi|^2\},\\
&S_b \overset{\text{def}}{=} \{\eta : \gamma_1|\xi|^2 \le |\xi-\eta|^2+|\eta|^2 \le \gamma_2|\xi|^2\}, \text{ and}\\
&S_c \overset{\text{def}}{=} \{\eta : \gamma_2|\xi|^2 \le |\xi-\eta|^2+|\eta|^2\}.
\end{aligned} \tag{2.2}$$

Thus we have

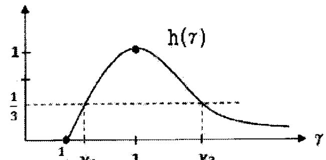

Figure 1:

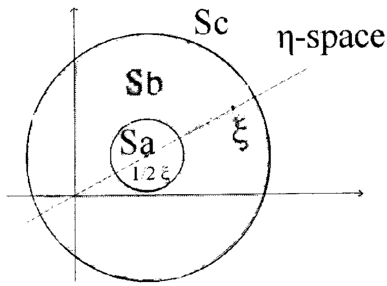

Figure 2:

$$
\begin{aligned}
&\left\|D^{\beta_0}S_+^{\beta_+}S_-^{\beta_-}\left(e^{-it\Delta}fe^{-it\Delta}g\right)\right\|_{L^2}\\
= &\iint \widehat{A}\left|\int \delta(\tau-|\xi-\eta|^2-|\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi \qquad (2.3)\\
= &\iint \widehat{A}\left|\int_{S_a}+\int_{S_b}+\int_{S_c}\delta(\tau-|\xi-\eta|^2-|\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi.
\end{aligned}
$$

Hence it is sufficient to bound each of the above integrals. For simplicity, we denote $\Phi(\eta)\equiv\tau-|\xi-\eta|^2-|\eta|^2$. Using Hölder inequality, we can bound the first integral in (2.3) as follows.

$$
\begin{aligned}
&\iint \widehat{A}\left|\int_{S_a}\delta(\tau-|\xi-\eta|^2-|\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi\\
\le &\iint \widehat{A}\int\frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}d\eta\int\delta(\Phi(\eta))\left||\xi-\eta|^{\alpha_1}\widehat{f}(\xi-\eta)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\eta d\tau d\xi\\
\lesssim &\iint\left\{\int\delta(\Phi(\eta))d\tau\right\}\left||\xi-\eta|^{\alpha_1}\widehat{f}(\xi-\eta)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\eta d\xi\le C\|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}},
\end{aligned}
$$

provided that

$$
\widehat{A}\int_{S_a}\frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta\le C,\quad \text{for all } \tau,\ \xi. \qquad (2.4)
$$

For the second integral we can get the desired bound in the same vain,

$$\iint \widehat{A}\left|\int_{S_b} \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi \le C\|f\|_{\dot{H}^{\alpha_1}}\|g\|_{\dot{H}^{\alpha_2}}, \tag{2.5}$$

provided that

$$\widehat{A}\int_{S_b} \frac{\delta(\Phi(\eta))}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \le C, \quad \text{for all } \tau,\ \xi. \tag{2.6}$$

Notice that we have $|\xi-\eta| \sim |\eta|$ and $|\xi-\varphi| \sim |\varphi|$ on the set S_c. Using the fact that $|z|^2 = z\bar{z}$, the Fubini theorem, and change of variables, we can bound the third integral in (2.3) as follows.

$$\begin{aligned}
&\iint \widehat{A}\left|\int_{S_c} \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\right|^2 d\tau d\xi \\
=\ &\iint \widehat{A}\int \delta(\Phi(\eta))\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta \int \delta(\Phi(\varphi))\overline{\widehat{f}}(\xi-\varphi)\overline{\widehat{g}}(\varphi)d\varphi d\tau d\xi \\
=\ &\iiint \widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))\widehat{f}(\xi-\eta)\widehat{g}(\eta)\overline{\widehat{f}}(\varphi)\overline{\widehat{g}}(\xi-\varphi)d\varphi d\eta d\xi \\
\lesssim\ &\left(\iiint \frac{\widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\left||\varphi|^{\alpha_1}\overline{\widehat{f}}(\varphi)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\varphi d\eta d\xi\right)^{1/2} \cdot \\
&\left(\iiint \frac{\widehat{A}\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\xi-\varphi||\xi-\eta|)^{\alpha_1+\alpha_2}}\left||\xi-\varphi|^{\alpha_1}\overline{\widehat{f}}(\xi-\varphi)|\xi-\eta|^{\alpha_2}\widehat{g}(\xi-\eta)\right|^2 d\varphi d\eta d\xi\right)^{1/2}.
\end{aligned}$$

Hence we have the following bound

$$\iint\int_{T_c} \widehat{A}\frac{\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}d\xi \left||\varphi|^{\alpha_1}\overline{\widehat{f}}(\varphi)|\eta|^{\alpha_2}\widehat{g}(\eta)\right|^2 d\varphi d\eta \le C\|f\|^2_{\dot{H}^{\alpha_1}}\|g\|^2_{\dot{H}^{\alpha_2}},$$

provided that

$$\int_{T_c} \widehat{A}\frac{\delta(\Phi(\eta)-\Phi(\xi-\varphi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\,d\xi \le C, \text{ for all } \varphi \text{ and } \eta, \tag{2.7}$$

where $T_c \overset{\text{def}}{=} \{\xi : |\xi-\varphi|^2+|\varphi|^2,\ |\xi-\eta|^2+|\eta|^2 \ge \gamma_2|\xi|^2\}$ and $\tau = |\xi-\varphi|^2+|\varphi|^2 = |\xi-\eta|^2+|\eta|^2$.

Therefore the proof of the Theorem is complete once the claims, (2.4), (2.6), and (2.7) are proved. □

Remark 1. *What left to be done are the following estimates. Claims: There is a constant C which is independent of τ, ξ, φ, and η such that the following inequalities hold.*

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \le C, \quad \textit{for all } \tau,\ \xi. \tag{2.8}$$

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \le C, \quad \textit{for all } \tau,\ \xi. \tag{2.9}$$

$$\int_{T_c} \widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \frac{\delta(|\xi-\varphi|^2+|\varphi|^2-|\xi-\eta|^2-|\eta|^2)}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\, d\xi \le C, \ \textit{for all } \varphi \textit{ and } \eta, \tag{2.10}$$

where $\tau = |\xi-\varphi|^2+|\varphi|^2 = |\xi-\eta|^2+|\eta|^2$. *The proofs of the above claims will be given in the next section.*

3 Proofs of Claims

Now we are ready to prove the claims. First we prove the claims which come from the proof of bilinear estimates for uv.

Lemma 1 (Claim (2.8)). *Let* $S_a = \{\eta : \frac{1}{2}|\xi|^2 \le |\eta|^2+|\xi-\eta|^2 \le \gamma_1|\xi|^2\}$. *If* $\beta_0+2\beta_++2\beta_-+\frac{n-2}{2} = \alpha_1+\alpha_2$ *and* $n \ge 2$, *then*

$$\widehat{D}^{2\beta_0} \widehat{S}_+^{2\beta_+} \widehat{S}_-^{2\beta_-} \int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \le C, \tag{3.1}$$

for all τ *and* ξ.

Proof. We set $\zeta \overset{\text{def}}{=} R\Big(\eta - \dfrac{\xi}{2}\Big)$, where R is the rotation such that $R\xi = |\xi|e_1$, then we have the indentities

$$|\xi-\eta| = \Big|\zeta - \frac{1}{2}|\xi|e_1\Big| \quad \text{and} \quad |\eta| = \Big|\zeta + \frac{1}{2}|\xi|e_1\Big|. \tag{3.2}$$

Then we use spherical coordinates

$$\zeta = (X_1, \cdots, X_n) \overset{\text{def}}{=} \rho(\cos\phi, \sin\phi\omega') \overset{\text{def}}{=} \rho\omega, \tag{3.3}$$

where $\omega \in S^n$ and $\omega' \in S^{n-1}$, so that we can rewrite the integral in (3.1) as

$$\int_{S_a} \frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta = \int \frac{1}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}} \frac{\rho_0^{n-1}}{|-4\rho_0|} d\omega, \tag{3.4}$$

where $\rho_0^2 = \frac{1}{4}(2\tau - |\xi|^2)$. Using the identity $d\omega = (\sin\phi)^{n-2} d\phi d\omega'$, the identities for $|\xi-\eta|$ and $|\eta|$ in (3.2), and the change of variables $p = \cos\phi$, we can simplify the integral further.

$$\iint \frac{1}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}} \rho_0^{n-2} \sin^{n-2}\phi d\phi d\omega \sim \frac{\rho_0^{n-2}}{\tau^{\alpha_1+\alpha_2}} \int_{-1}^{1} \frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}} dp, \tag{3.5}$$

where $\lambda = \dfrac{2|\xi|\rho_0}{\tau} = \dfrac{|\xi|\sqrt{2\tau-|\xi|^2}}{\tau}$. Notice that $0 \le \lambda \le \dfrac{1}{3}$ under the restriction $\tau = |\xi-\eta|^2+|\eta|^2$ for $\eta \in S_a$.

We set $\tau \overset{\text{def}}{=} \gamma|\xi|^2$ which implies that $\lambda = \dfrac{\sqrt{2\gamma-1}}{\gamma} = h(\gamma)$. Then we have $\dfrac{1}{2} \le \gamma \le \gamma_1$ which implies that $\tau \sim |\xi|^2$, and $\rho_0 = \frac{1}{2}\sqrt{2\gamma-1}|\xi| \le |\xi|$. Thus we can estimate the quantity in (3.1) as follows.

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_a}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta$$
$$\le |\xi|^{2\beta_0+4\beta_++4\beta_--2\alpha_1-2\alpha_2}\Big||\gamma|-1\Big|^{2\beta_-}\left(\frac{\sqrt{2\gamma-1}|\xi|}{2}\right)^{n-2}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp.$$

The above quantity is bounded if we require that $n \ge 2$ and $\beta_0 + 2\beta_+ + 2\beta_- + \dfrac{n-2}{2} = \alpha_1 + \alpha_2$. □

Lemma 2 (Claim (2.9)). *Let* $S_b = \{\eta : \gamma_1|\xi|^2 \le |\eta|^2 + |\xi-\eta|^2 \le \gamma_2|\xi|^2\}$. *If*

$$\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,$$

$$n \ge 2, \quad \beta_- \ge \alpha_j - \frac{n-1}{2}, \quad \beta_- \ge 0, \quad \textit{and} \quad (\beta_-, \alpha_j) \ne (0, \frac{n-1}{2}),$$

for $j = 1, 2$, *then*

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \le C, \tag{3.6}$$

for all τ *and* ξ.

Proof. As in the proof of Lemma 1, we set $\zeta \overset{\text{def}}{=} R\Big(\eta - \dfrac{\xi}{2}\Big)$, where the rotation $R\xi = |\xi|e_1$, and

$$\zeta = (X_1, \cdots, X_n) \overset{\text{def}}{=} \rho(\cos\phi, \sin\phi\omega') \overset{\text{def}}{=} \rho\omega. \tag{3.7}$$

Using the above, the identity $d\omega = (\sin\phi)^{n-2}d\phi d\omega'$, and the identities for $|\xi-\eta|$ and $|\eta|$, we can rewrite the integral in (3.6) as

$$\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta \sim \frac{\rho_0^{n-2}}{\tau^{\alpha_1+\alpha_2}}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp, \tag{3.8}$$

where $\rho_0^2 = \dfrac{1}{4}(2\tau - |\xi|^2)$, the change of variables $p = \cos\phi$, and $\lambda = \dfrac{2|\xi|\rho_0}{\tau} = \dfrac{|\xi|\sqrt{2\tau-|\xi|^2}}{\tau}$.

Again we set $\tau \overset{\text{def}}{=} \gamma|\xi|^2$ and then we have $\gamma_1 \le \gamma \le \gamma_2$ and $\dfrac{1}{3} \le \lambda \le 1$ under the restriction $\tau = |\xi-\eta|^2 + |\eta|^2$ for $\eta \in S_b$. These imply that

$$\tau \sim |\xi|^2, \quad (\gamma-1)^2 \sim (1-\lambda), \quad \text{and} \quad \rho_0 = \frac{1}{2}\sqrt{2\gamma-1}|\xi| \sim |\xi|. \tag{3.9}$$

Now we can combine the above observations to simplify the quantity in (3.1) as follows.

$$\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-}\int_{S_b}\frac{\delta(\tau-|\xi-\eta|^2-|\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\,d\eta$$

$$\sim|\xi|^{2\beta_0+4\beta_+ +4\beta_- +n-2-2\alpha_1-2\alpha_2}\Big|\gamma-1\Big|^{2\beta_-}\int_{-1}^{1}\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp.$$

To bound the above integral we split it into two parts, one is over [-1, 0] while the other is over [0, 1]. Since the estimates for the two parts are the same, thus we only prove the second part. First we have

$$\int_0^1\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp\sim\int_0^1\frac{(1-p)^{(n-3)/2}}{(1-\lambda p)^{\alpha_2}}dp \tag{3.10}$$

Using the change of variables $p=-(1-\lambda)q+\lambda$, the integral is changed into

$$(1-\lambda)^{\frac{n-3}{2}+1-\alpha_2}\int_{-1}^{\frac{\lambda}{1-\lambda}}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq.$$

Again we split the above integral into two parts. For the first part, we have

$$\int_{-1}^{1}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq\le C,$$

provided that $n\geq 2$. For the second part, we have

$$\int_1^{\frac{\lambda}{1-\lambda}}\frac{(1+q)^{(n-3)/2}}{(1+\lambda+\lambda q)^{\alpha_2}}\,dq\sim\begin{cases}(1-\lambda)^{-\frac{n-1}{2}+\alpha_2} & \text{for}\quad \alpha_2<\frac{n-1}{2},\\ |\log(1-\lambda)| & \text{for}\quad \alpha_2=\frac{n-1}{2},\\ C\quad\text{for}\quad \alpha_2>\frac{n-1}{2}, & \end{cases}$$

Thus we get

$$(1-\lambda)^{\beta_-}\int_0^1\frac{(1-p^2)^{(n-3)/2}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}dp$$

$$\sim C(1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}+\begin{cases}(1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}(1-\lambda)^{-\frac{n-1}{2}+\alpha_2} & \text{for}\quad \alpha_2<\frac{n-1}{2},\\ (1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}|\log(1-\lambda)| & \text{for}\quad \alpha_2=\frac{n-1}{2},\\ (1-\lambda)^{\beta_-+\frac{n-1}{2}-\alpha_2}C & \text{for}\quad \alpha_2>\frac{n-1}{2},\end{cases}$$

Hence the above integral is bounded if (β_-,α_2) is in the set $S_1\cap(S_2\cup S_3\cup S_4)$, where

$$S_1\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_-\geq\alpha_2-\frac{n-1}{2}\},\quad S_2\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_-\geq 0,\alpha_2<\frac{n-1}{2}\},$$

$S_3\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_->\alpha_2-\frac{n-1}{2},\alpha_2=\frac{n-1}{2}\}$, and $S_4\stackrel{\text{def}}{=}\{(\beta_-,\alpha_2):\beta_-\geq\alpha_2-\frac{n-1}{2},\alpha_2>\frac{n-1}{2}\}$.

Therefore the quantity in (3.6) is bounded if we require that

$$\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,$$

$$n \geq 2, \quad \beta_- \geq \alpha_2 - \frac{n-1}{2}, \quad \beta_- \geq 0, \quad \text{and} \quad (\beta_-, \alpha_2) \neq (0, \frac{n-1}{2}).$$

The conditions for α_1 is the same as that of α_2. This completes the proof. $\square$

Remark 2. *Notice that for the integral over S_c we have $0 \leq \lambda \leq \frac{1}{3}$, $\gamma_2 \leq \gamma$, $\tau = \gamma|\xi|^2$, and $\rho_0 = \sqrt{2\gamma - 1}|\xi|/2 \sim \sqrt{\gamma}|\xi|$. If we follow the same path to estimates it, then we obtain*

$$\begin{aligned} &\widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi-\eta|^2 - |\eta|^2)}{|\xi-\eta|^{2\alpha_1}|\eta|^{2\alpha_2}}\, d\eta \\ \sim\ & |\xi|^{2\beta_0+4\beta_++4\beta_-+n-2-2\alpha_1-2\alpha_2}\gamma^{2\beta_++2\beta_-+\frac{n-2}{2}-\alpha_1-\alpha_2} \int_{-1}^{1} \frac{(1-u^2)^{\frac{n-3}{2}}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}}\, dp \leq C, \end{aligned} \tag{3.11}$$

provided that $\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2$, $n \geq 2$, and $2\beta_+ + 2\beta_- + \frac{n-2}{2} - \alpha_1 - \alpha_2 \leq 0$. The last condition implies that $\beta_0 \geq 0$ which we shall see that this is not good enough.

Lemma 3 (Claim (2.10)). *Let $T_c(\eta, \varphi) \overset{\text{def}}{=} \{\xi : |\xi-\eta|^2 + |\eta|^2,\ |\xi-\varphi|^2 + |\varphi|^2 \geq \gamma_2|\xi|^2\}$. If $\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2$ and $\beta_0 > -\frac{n-1}{2}$, then*

$$\int_{T_c} \widehat{D}^{2\beta_0}\widehat{S}_+^{2\beta_+}\widehat{S}_-^{2\beta_-} \frac{\delta(|\xi-\varphi|^2 + |\varphi|^2 - |\xi-\eta|^2 - |\eta|^2)}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi \leq C, \tag{3.12}$$

where $\tau = |\xi-\varphi|^2 + |\varphi|^2 = |\xi-\eta|^2 + |\eta|^2$.

Proof. Let $\Phi(\xi) \overset{\text{def}}{=} |\xi-\varphi|^2 + |\varphi|^2 - |\xi-\eta|^2 - |\eta|^2$ and $P(\varphi,\eta) \overset{\text{def}}{=} \{\xi : \Phi(\xi) = 0\}$. Since $|\xi-\varphi|^2 + |\varphi|^2 \geq \gamma_2|\xi|^2$ and $\gamma_2 > 16$, thus we have $|\varphi| + |\xi-\varphi| \geq 4|\xi|$ and analogously we have $|\eta| + |\xi-\eta| \geq 4|\xi|$.

Using triangle inequality, we get

$$\frac{3}{5}|\eta| \leq |\xi-\eta| \leq \frac{5}{3}|\eta| \quad \text{and} \quad \frac{3}{5}|\varphi| \leq |\xi-\varphi| \leq \frac{5}{3}|\varphi|,$$

and then

$$|\xi| \leq \frac{2}{3}\min\{|\eta|,\ |\xi-\eta|,\ |\varphi|,\ |\xi-\varphi|\}.$$

On the plane P we have $|\xi-\varphi|^2 + |\varphi|^2 = |\xi-\eta|^2 + |\eta|^2$ which implies that

$$|\xi-\varphi| \sim |\varphi| \sim |\xi-\eta| \sim |\eta|.$$

Set $(\xi-\eta)\cdot(-\eta) \overset{\text{def}}{=} |\xi-\eta||\eta|\cos\theta$, through some calculations we can show that

$$\cos\theta \geq \cos\theta_0 = \frac{\sqrt{5}}{3} > \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}.$$

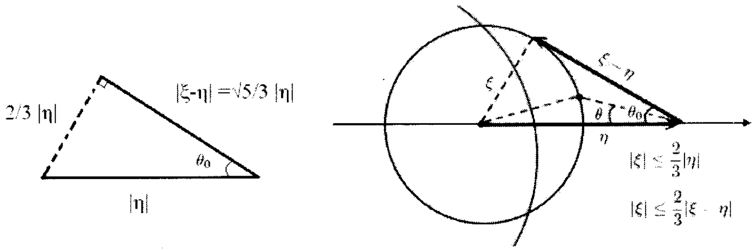

Figure 3:

Hence the angle between $-\eta$ and $\xi-\eta$ is restricted on $0 \le \theta \le \theta_0 \le \frac{\pi}{4}$.

Without loss of generality, we assume $|\varphi| > |\eta|$. Follow the idea used in [FK], we decompose $S^{n-2} = \bigcup_{j=1}^{N} \Omega_j$, where Ω_j are disjoint and the angle between any two unit vectors lie in the same Ω_j is less than θ_0, and N is a finite integer. Denote

$\Gamma_j \stackrel{\text{def}}{=} \left\{\xi \in \mathbb{R}^n/\{0\} : \frac{\xi}{|\xi|} \in \Omega_j\right\}$, $\chi_j \stackrel{\text{def}}{=}$ characteristic function of Γ_j, $f_j \stackrel{\text{def}}{=} \chi_j f$, and $g_j \stackrel{\text{def}}{=} \chi_j g$.

Thus we have $f = \sum_{j=1}^{N} f_j$ and $g = \sum_{j=1}^{N} g_j$. Then we can split the integral into finitely many pieces,

$$
\begin{aligned}
&\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\, d\eta \right\|_{L^2(\tau\ge 16|\xi|^2)} \\
\le\ & \sum_{j,k} \left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\widehat{f_j}(\xi-\eta)\widehat{g_k}(\eta)\, d\eta \right\|_{L^2(\tau\ge 16|\xi|^2,\ \theta_0<\pi/4)}.
\end{aligned}
$$

There exists a cone Γ with an aperture $2\theta_0$ such that $\eta \in \Gamma_k \subset \Gamma$ and $\xi - \eta \in \Gamma_j \subset -\Gamma$.

$$
\begin{aligned}
&\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi-\eta|^2 - |\eta|^2)\widehat{f_j}(\xi-\eta)\widehat{g_k}(\eta)\, d\eta \right\|^2_{L^2(\tau\ge 16|\xi|^2)} \\
=\ & \iiint D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-} \delta(\Phi(\xi))\widehat{f_j}(\xi-\eta)\widehat{g_k}(\eta)\bar{\widehat{f_j}}(\varphi)\bar{\widehat{g_k}}(\xi-\varphi)\, d\varphi\, d\eta\, d\xi,
\end{aligned}
$$

where $\eta \in \Gamma_k \subset \Gamma$, $\xi-\eta \in \Gamma_j \subset -\Gamma$, $\varphi \in \Gamma_j \subset -\Gamma$, and $\xi - \varphi \in \Gamma_k \subset \Gamma$. Through elementary argument, we have the following identity, see [H],

$$
\int D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-} \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi = \int \frac{D^{2\beta_0} S_+^{2\beta_+} S_-^{2\beta_-}}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} \frac{d\mu}{|\nabla\Phi(\xi)|}, \tag{3.13}
$$

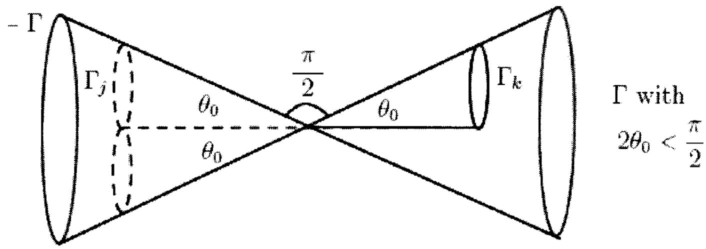

Figure 4:

where $d\mu$ is the surface measure on the surface $\{\xi : \Phi(\xi) = 0\}$. The facts $\xi - \varphi \in \Gamma$ and $\xi - \eta \in -\Gamma$ imply that $|\nabla\Phi(\xi)| \sim |\varphi|$ since

$$|\nabla\Phi(\xi)| = |-2(\varphi - \eta)| = 2\sqrt{|\varphi|^2 + |\eta|^2 - 2\varphi \cdot \eta} \sim |\varphi|.$$

Let ξ' be the projection of ξ onto the plane P, see figure 5,

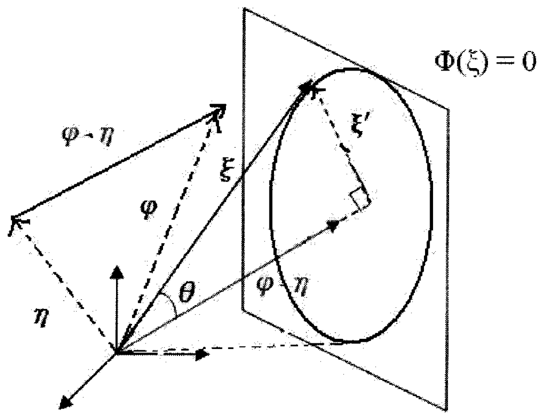

Figure 5:

$$\xi' \stackrel{\text{def}}{=} \xi - \xi \cdot \frac{\varphi - \eta}{|\varphi - \eta|} \frac{\varphi - \eta}{|\varphi - \eta|},$$

and $P_{\frac{\varphi+\eta}{2}}$ the projection of $\frac{\varphi + \eta}{2}$ onto the plane P,

$$P_{\frac{\varphi+\eta}{2}} \stackrel{\text{def}}{=} \frac{\varphi + \eta}{2} - \frac{\varphi + \eta}{2} \cdot \frac{\varphi - \eta}{|\varphi - \eta|} \frac{\varphi - \eta}{|\varphi - \eta|}.$$

Denote the rotation taking $\varphi - \eta$ to $|\varphi - \eta| e_1$ by R and the change of coordinates

$$\nu \stackrel{\text{def}}{=} R\left(\xi - \frac{\varphi + \eta}{2}\right) \stackrel{\text{def}}{=} (X_1, X_2, \cdots, X_n).$$

Thus we get $|\xi'| \le |\xi|$ and

$$\begin{aligned} R\left(\xi' - P_{\frac{\varphi+\eta}{2}}\right) &= R\left(\xi - \frac{\varphi+\eta}{2}\right) - R\left(\xi - \frac{\varphi+\eta}{2}\right)\cdot R\left(\frac{\varphi-\eta}{|\varphi-\eta|}\right) R\left(\frac{\varphi-\eta}{|\varphi-\eta|}\right) \\ &= \nu - \nu\cdot e_1\, e_1 = (X_1, X_2, \cdots, X_n) - (X_1, 0, \cdots, 0) = (0, X_2, \cdots, X_n), \end{aligned}$$

and then $d\xi' = dX_2 \cdots dX_n = d\mu$. Hence we obtain $\delta(\Phi(\xi))d\xi = \dfrac{d\mu}{|\nabla\Phi(\xi)|} \sim \dfrac{d\xi'}{|\varphi|}$.

Therefore we can now bound the integral (3.13) as follows.

$$\begin{aligned} &\int_{P(\eta,\varphi),\,|\xi|\le\frac{2}{3}|\eta|} |\xi|^{2\beta_0}\left||\xi-\varphi|^2+|\varphi|^2+|\xi|^2\right|^{2\beta_+}\left||\xi-\varphi|^2+|\varphi|^2-|\xi|^2\right|^{2\beta_-}\frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}}\,d\xi \\ &\sim |\varphi|^{4\beta_++4\beta_--2\alpha_1-2\alpha_2}\int_{|\xi|<|\varphi|}|\xi|^{2\beta_0}\frac{d\xi'}{|\varphi|}. \end{aligned}$$

If $\beta_0 \ge 0$, then we get the bound

$$\int_{|\xi|<|\varphi|} |\xi|^{2\beta_0}\, d\xi' \le \int_{|\xi|\le|\varphi|} |\varphi|^{2\beta_0}\, d\xi' \sim |\varphi|^{2\beta_0+n-1}.$$

If $\beta_0 < 0$, then we get the bound

$$\int_{|\xi'|\le|\varphi|} |\xi|^{2\beta_0}\, d\xi' \le \int_{|\xi'|\le|\varphi|} |\xi'|^{2\beta_0}\, d\xi' \sim |\varphi|^{2\beta_0+n-1},$$

provided that $2\beta_0 + n - 1 > 0$. Finally combining the above results we have

$$(3.13) \lesssim |\varphi|^{4\beta_++4\beta_--2\alpha_1-2\alpha_2+2\beta_0+n-2} \le C,$$

provided that $2\beta_+ + 2\beta_- + \beta_0 + \frac{n-2}{2} = \alpha_1 + \alpha_2$ and $2\beta_0 + n - 1 > 0$. This completes the proof. □

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