APRIORI ESTIMATES FOR THE 2-D WAVE EQUATION

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Chapter 1, Introduction :

In the present work we will be concerned with certain regularity properties of the linear Klein-Gordon equation. The motivation for this questions comes from certain nonlinear systems of wave equations, see [2]. In order to formulate the problem let us start by considering the Klein-Gordon Equation

$$\Box\phi + m^2\phi = 0, \qquad (0.1a)$$

with initial data given at some fixed time, for example $t = 0$, by

$$\phi(0,x) = 0 \quad , \quad \phi_{,t}(0,x) = f(x). \qquad (0.1b)$$

The conservation of energy, to be explained below, dictates as a natural hypothesis $f \in L^2(R^2)$. One can write the solution of the equation above in integral form as follows

$$\phi(t,x) = \int_{R^2} R(t, x-y) f(y) dy \quad , \qquad (0.2a)$$

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where $R(t,x)$ is the Riemann function, which in this case can be computed explicitly.

$$R(t,x) := \frac{1}{2\pi} \frac{\cos\left[m\sqrt{t^2-|x|^2}\right]}{\sqrt{t^2-|x|^2}} H(t^2-|x|^2) \quad , \tag{0.2b}$$

where $H(t)$ is the Heaviside function. Notice that $\sqrt{t^2-|x|^2}$ is the hyperbolic distance between the point $(t,x) \in R \times R^2$ and the origin with respect to the Minkowski metric $g_{\mu\nu} := diag(-1,1,1)$. In the present paper we would like to consider the solution $\phi(t,x)$ with rough initial data, namely $f \in L^2(R^2)$. From the expression in (0.2a) it is obvious that $\phi(t,x)$ just fails to be in L^∞. Standard estimates however imply that ϕ is in the class of bounded mean oscillation functions, an idea introduced by F. John and L. Nirenberg, [12], henceforth to be denoted by BMO. Functions in BMO have the property that, on every cube, they can be approximated in the L mean by their average, with an error independent of the cube, see [7]. Roughly speaking BMO functions can have logarithmic type of singularities and there is no reason why the square of the function is still in BMO. Here we will consider a square expression, namely $\phi^2(t,x)$ and see that the L^2 norm in space of the first deivatives is bounded by a BMO function. It is worthwhile to observe here that if one assumes that $f \in B_2^{0,1}(R^2)$, which is the Besov space with indices $(2,1)$, then the solution is in L^∞.

There are two standard estimates one can derive for the equation (0.1a), one is the energy estimate which follows from the conservation laws

$$\partial_\mu T^\mu_\nu = 0 \quad . \tag{0.3a}$$

$T_{\mu\nu}$ is the energy momentum tensor defined by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi + m^2\phi^2\right) \quad . \tag{0.3b}$$

The quantity $T_{0,0} = 1/2(|\nabla\phi|^2 + \phi_{,t}^2 + m^2\phi^2)$ is positive and integrating over space-time we have the bound

$$\int_{R^2} \left[|\nabla\phi|^2 + \phi_{,t}^2 + m^2\phi^2\right] dx = \text{const.} \tag{0.4}$$

The Sobolev inequality implies that for every fixed time $\phi(t,\cdot)$ is a BMO function.

The other estimate, due originally to Strichartz [4], reads

$$\|D^{1/2}\phi\|_{L^6(R^3)} \leq C\|f\|_{L^2(R^2)} \quad , \tag{0.5}$$

where $D^{1/2}$ is half a derivative defined via the Fourier transform. Notice that the left hand side of the above inequality implies again ,via the Sobolev inequality, that ϕ is a BMO function but over space-time. Estimate (0.5) is derived using the Fourier transform and we will use these ideas in the present work. As a matter of fact the estimate of Strichartz will follow from one of the theorems that we will present later.

The solution of (0.1) can be written

$$\phi(t,x) = \int_{R^2} \sin(\omega(\xi)t)e^{ix\cdot\xi}\hat{f}(\xi)d\mu(\xi) \tag{0.6a}$$

$$\omega(\xi) := \sqrt{|\xi|^2+m^2} \qquad \text{and} \qquad d\mu(\xi) := \frac{d\xi}{\omega(\xi)} \tag{0.6b}$$

Now squaring the above expression gives $\phi^2(t,x)$ as a combination of two terms

$$A := \int_{R^2\times R^2} \cos\left[(\omega(\xi_1)+\omega(\xi_2))t\right] e^{ix\cdot(\xi_1+\xi_2)}\hat{f}(\xi_1)\hat{f}(\xi_2)d\mu(\xi_1)d\mu(\xi_2)$$

$$B := \int_{R^2\times R^2} \cos\left[(\omega(\xi_1)-\omega(\xi_2))t\right] e^{ix\cdot(\xi_1+\xi_2)}\hat{f}(\xi_1)\hat{f}(\xi_2)d\mu(\xi_1)d\mu(\xi_2) \quad .$$

¿From the above considerations one can see that the square expression is built from the basic integral operator

$$E(f)(t,x) := \int_{R^2} \exp\{i(\omega(\xi)t + x\cdot\xi)\}\hat{f}(\xi)d\mu(\xi) \tag{0.7a}$$

$$\omega(\xi) := \sqrt{|\xi|^2+m^2} \qquad \text{and} \qquad d\mu(\xi) := \frac{d\xi}{\omega(\xi)} \quad . \tag{0.7b}$$

The measure $d\mu(\xi)$ is a natural object, it is the Leray measure on the Hyperbola defined by the equation

$$\left\{(\tau,\xi)\in R\times R^2 \quad : \quad \tau=\sqrt{|\xi|^2+m^2}\right\} \quad .$$

¿From the operator $E(f)$ one can form two types of squares

$$\Phi_i(t,x):=E(f)E(g)(t,x) \qquad \text{and} \qquad \Phi_o(t,x):=E(f)\bar{E}(g)(t,x) \quad . \tag{0.8}$$

We will see that in order to prove estimates for $\phi^2(t,x)$ we have to give a proof for both $\Phi_{i,o}$ mentioned above.

Now let us state the first theorem.

Theorem 0.1 : *Consider the solution of equation (0.1a,b). Let* $D_{t,x}$ *denote any first order derivative and call*

$$e(t):=\left\{\int_{R^2}|D_{t,x}\phi^2(t,x)|^2dx\right\}^{\frac{1}{2}} \quad . \tag{0.9a}$$

Then there is a positive function $b(t)$ *such that*

$$e(t)\le b(t) \quad , \tag{0.9b}$$

where

$$\|b\|_{bmo}\le C\|f\|_{L^2}^2 \quad , \tag{0.9c}$$

and the bmo norm is a local version of BMO.

Before we proceed we would like to explain the BMO and bmo norms as well as an important duality property of these spaces, see [6], [7] , [10]. Let us start by calling

$$(g)_I:=\frac{1}{|I|}\int_I g(x)dx$$

the average of a function over an arbitrary cube with edges parallel to the coordinate axis. The BMO norm is defined by

$$\|g\|_{BMO}:=\sup_I\frac{1}{|I|}\int_I|g(x)-(g)_I|dx \quad . \tag{0.10}$$

The supremum above is taken over all cubes. The *bmo* norm is the following modification, see [6] , [10]

$$\|g\|_{bmo} := \sup_{|I|\le 1} \frac{1}{|I|} \int\limits_I |g-(g)_I|dx + \sup_{|I|\ge 1} \frac{1}{|I|} \int\limits_I |g|dx \quad . \tag{0.11}$$

There is an important duality property for this space, namely

$$\left(h^1\right)^* = bmo \quad , \tag{0.12a}$$

which in practical terms means the following

$$\left| \int\limits_{R^n} fg dx \right| \le \|f\|_{h^1} \|g\|_{bmo} \quad . \tag{0.12b}$$

The space h^1 is again a localized version of the Hardy space H^1. It can be described by giving the norm of the space, [9] , [10]. Consider first a smooth function $\eta(x)$ supported in the unit ball of R^n with the property $\int_{R^n} \eta(x)dx = 1$. Given an integrable function f, form the following average

$$M_\epsilon[f](x_0) := \frac{1}{\epsilon^n} \left| \int\limits_{R^n} \eta\left(\frac{x_0 - x}{\epsilon}\right) f(x)dx \right|$$

and set

$$f^*(x_0) := \sup_{0<\epsilon<\infty} M_\epsilon[f](x_0) \quad .$$

The usual Hardy space is defined by

$$H^1 := \left\{ f \in L^1(R^n) \quad \text{and} \quad f^* \in L^1(R^n) \right\}$$

Now form, for a function $f \in L^1_{loc}(R^n)$

$$f^{**}(x_0) := \sup_{0<\epsilon<1} M_\epsilon[f](x_0) \quad .$$

The local version of the Hardy space, introduced by Goldberg [6], is

$$h^1 := \left\{ f \in L^1(R^n) \quad \text{and} \quad f^{**} \in L^1(R^n) \right\} \quad .$$

For a function $f \in h^1(R^n)$ we set, see [9] , [10]

$$\|f\|_{h^1(R^n)} := \|f^{**}\|_{L^1(R^n)} \quad .$$

There is a natural estimate for the operator $E(f)$ defined in (0.7). It reads

$$\|E(f)\|_{L^6(R^3)} \leq C\|\hat{f}\|_{L^2(d\mu)} \quad . \tag{0.14a}$$

The right hand side of the above is the L^2 norm defined via the Leray measure on the Hyperbola $d\mu := d\xi/\omega(\xi)$. To be specific

$$\|\hat{f}\|_{L^2(d\mu)} := \left\{ \int_{R^2} |\hat{f}(\xi)|^2 \frac{d\xi}{\omega(\xi)} \right\}^{1/2} \quad . \tag{0.14b}$$

This is essentially the same as estimate (0.5) and it is the estimate derived by Strichartz. It is optimal as stated, however there is a smoothing phenomenon. One quarter of the derivative applied to $E^2(f)$ is bounded in an appropriate sense by the same right hand side. The theorem can be stated as follows.

Theorem 0.2 : *Consider the operator $E(f)(t,x)$ defined in (0.7) and the square expressions*

$$\Phi_i(t,x) := E(f)E(g) \qquad \text{and} \qquad \Phi_o(t,x) := E(f)\bar{E}(g) \quad . \tag{0.15a}$$

The following mixed space-time estimate holds

$$\left\{ \int_R \left[\int_{R^2} |D^{1/4}\Phi_{i,o}|^2 dx \right]^{4/3} dt \right\}^{3/8} \leq C\|\hat{f}\|_{L^2(d\mu)}\|\hat{g}\|_{L^2(d\mu)} \quad , \tag{0.15b}$$

where $D^{1/4}$ denotes one quarter of the derivative taken via the Fourier transform.

A final theorem is concerned with products $E(f)E(g)$ or $E(f)\bar{E}(g)$ when the smoothness of f and g is not the same, this is a refinement of Theorem 0.1.

Theorem 0.3 : *Assume that ϕ and ψ are solutions of the equations*

$$\Box\phi + m^2\phi = 0 \qquad \Box\psi + m^2\psi = 0$$

with initial data

$$\phi(0,x) := 0 \quad \phi_{,t}(0,x) := f(x) \qquad , \qquad \psi(0,x) := 0 \quad \psi_{,t}(0,x) := g(x) \quad .$$

Then

$$\left\| (||\tau| - |\xi|| + m)^{1/2}\, \hat{\phi} * \hat{\psi}(\tau,\xi) \right\|_{L^2(R^3)} \leq C\|\hat{f}/\omega\|_{L^2}\|\hat{g}/\|_{L^2} \quad , \tag{0.16a}$$

where $\omega := \sqrt{|\xi|^2 + m^2}$. Moreover the quantity defined bellow

$$\epsilon(t) := \left\{ \int_{R^2} |\phi\psi|^2 dx \right\}^{1/2} \tag{0.16b}$$

satisfies the estimate

$$\epsilon(t) \leq b(t) \quad , \tag{0.16c}$$

where

$$\|b\|_{bmo} \leq C \left\| \hat{f}/\omega \right\|_{L^2} \|\hat{g}\|_{L^2} \quad . \tag{0.16d}$$

Notice that in the estimate above ϕ can only satisfy the estimate

$$\sup_t \|\phi(t,\cdot)\|_{L^2(R^2)} \leq \|\hat{f}/\omega\|_{L^2(R^2)}$$

while ψ for each fixed time belongs to H^1. In what follows we will use $\hat{}$ to denote the Fourier transform of a function over the space-time variables. We will use $\tilde{}$ to denote the partial Fourier transform over the space variables and we will always denote

$$\omega(\xi) =: \sqrt{|\xi|^2 + m^2} \quad .$$

Chapter 2, Demonstration :

We will start by considering the integral operator defined in (0.7), namely

$$E(f)(t,x) := \int_{R^2} \exp[i(t\sqrt{|\xi|^2+m^2} + x\cdot\xi)]\hat{f}(\xi)\frac{d\xi}{\sqrt{|\xi|^2+m^2}} \quad . \qquad (1.1a)$$

In what follows we will denote

$$\omega(\xi) := \sqrt{|\xi|^2+m^2} \quad . \qquad (1.1b)$$

Notice that the Fourier transform of the expression above is

$$\hat{E}(f)(\tau,\xi) = \delta(\tau^2 - |\xi|^2 - m^2)H(\tau)\hat{f}(\xi)d\mu(\xi)$$

where

$$d\mu(\xi) := d\xi/\omega(\xi) = d\xi/\tau$$

is the Leray measure on the hyperbola defined by $\{\tau = \omega(\xi)\}$. The leray measure can be explained as follows. Consider the function $S := \tau^2 - |\xi|^2$ and write

$$d\tau d\xi = d\mu_S dS$$

the measure $d\mu_S$ is a natural object defined on the surface $\{S = s\}$. Consider the following quadratic expressions

$$\hat{\Phi}_i(\tau,\zeta) := \hat{E}(f) * \hat{E}(g) \qquad (1.2a)$$

$$\hat{\Phi}_o(\tau,\zeta) := \hat{E}(f) * \hat{\bar{E}}(g) \quad . \qquad (1.2b)$$

Notice that these are the Fourier transforms of the quantities

$$E(f)E(g) \qquad \text{and} \qquad E(f)\bar{E}(g)$$

respectively. The idea of the estimate in Theorem 0.1 is very simple. It is based on the observation

$$\left\|(||\tau|-|\zeta||+m)^{1/2}\,\hat{\Phi}_{i,o}(\tau,\zeta)\right\|_{L^2(R^3)} \leq C\|\hat{f}\|_{L^2(R^2)}\|\hat{g}\|_{L^2(R^2)} \quad . \qquad (A)$$

To make this more precise and to prepare for Theorems 0.2,3 we need the following technical lemma.

Lemma 1.1 : *With the above notation we have the estimates*

$$(|\zeta|+|\tau|)\hat{\Phi}_{i,o}(\tau,\zeta) \sim \hat{K}_{i,o}(\tau,\zeta)\hat{D}_{i,o}(\tau,\zeta) \qquad (1.3)$$

where

$$\hat{K}_i(\tau,\zeta) := \frac{\tau^{1/2}H(\tau^2-|\zeta|^2-4m^2)H(\tau)}{[(\tau^2-|\zeta|^2)^2+4m^2|\zeta|^2]^{\frac{1}{4}}} \qquad (1.4a)$$

$$\hat{K}_o(\tau,\zeta) := \frac{|\zeta|^{\frac{1}{2}}(|\zeta|^2-\tau^2)^{\frac{1}{4}}H(|\zeta|^2-\tau^2)}{\{[(|\zeta|^2-\tau^2)^2+4m^2|\zeta|^2]\,(|\zeta|^2-\tau^2+4m^2)\}^{\frac{1}{4}}} \qquad (1.4b)$$

and the quantities $\hat{D}_{i,o}$ satisfy the bounds

$$\|\hat{D}_{i,o}\|_{L^2(R^3)} \leq C\|f\|_{L^2(R^2)}\|g\|_{L^2(R^2)} \quad , \qquad (1.5)$$

where $H(s)$ denotes the Heaviside function.

Proof :

The proof is based on an explicit computation of the quantities $\hat{\Phi}_{i,o}$. Let us compute first the quantity $\hat{\Phi}_o$

A) Computation of $\hat{\Phi}_o(\tau,\zeta)$:

$$E(f)\bar{E}(g)(t,x) = \int_{R^4} \exp(i\Omega)\hat{f}(\xi)\bar{\hat{g}}(-\eta)\frac{d\xi d\eta}{\omega(\xi)\omega(\eta)} \qquad (1.6a)$$

where

$$\Omega := t(\omega(\xi)-\omega(\eta)) + x\cdot(\xi+\eta) \quad . \qquad (1.6b)$$

Call $\tau := \omega(\xi) - \omega(\eta)$ and set $\zeta := \xi + \eta$ and $X := \xi - \eta$. If we write $X = (X_1, X_2)$ where X_1 is in the direction of ζ then X satisfies the equation

$$\frac{|\zeta|^2 - \tau^2}{|\zeta|^2 - \tau^2 + 4m^2} \frac{X_1^2}{\tau^2} - \frac{X_2^2}{|\zeta|^2 - \tau^2 + 4m^2} = 1 \quad . \tag{1.7a}$$

Notice that $|\zeta|^2 - \tau^2 \geq 0$. Introducing natural coordinates on the hyperbola defined by (1.7a)

$$X_1 := a \cosh \alpha \qquad a := \frac{\tau \sqrt{|\zeta|^2 - \tau^2 + 4m^2}}{\sqrt{|\zeta|^2 - \tau^2}} \tag{1.7b}$$

$$X_2 := b \sinh \alpha \qquad b := \sqrt{|\zeta|^2 - \tau^2 + 4m^2} \tag{1.7c}$$

we can compute the Jacobian of the transformation $dX_1 dX_2 = J d\tau d\alpha$

$$J(\tau, \zeta, \alpha) = \frac{|\zeta|^2(|\zeta|^2 - \tau^2 + 4m^2)\cosh^2\alpha - \tau^2(|\zeta|^2 - \tau^2)}{(|\zeta|^2 - \tau^2)^{\frac{3}{2}}} \quad . \tag{1.8a}$$

On the other hand a straightforward calculation gives

$$4\omega(\xi)\omega(\eta) = \frac{|\zeta|^2(|\zeta|^2 - \tau^2 + 4m^2)\cosh^2\alpha - \tau^2(|\zeta|^2 - \tau^2)}{(|\zeta|^2 - \tau^2)} \quad . \tag{1.8b}$$

Combining (1.8a,b) gives the formula

$$\hat{\Phi}_o(\tau, \zeta) = \frac{1}{\sqrt{|\zeta|^2 - \tau^2}} \int_R \hat{f}\left(\frac{\zeta + X(\alpha)}{2}\right) \bar{\hat{g}}\left(\frac{X(\alpha) - \zeta}{2}\right) d\alpha \tag{1.9a}$$

where the vector $X(\alpha)$ can be expressed in a simple manner using complex notation

$$\zeta := |\zeta| e^{i\phi} \quad \text{and} \quad X(\alpha) = e^{i\phi}\left[a(\tau, \zeta)\cosh\alpha + i b(\tau, \zeta)\sinh\alpha\right] \tag{1.9b}$$

where a, b are defined in (1.7b,c) . We will need formula (1.9) later. Now the Lemma follows from the estimate

$$\frac{1}{|\zeta|^2 - \tau^2} \int_R \frac{d\alpha}{J} \sim \frac{H(|\zeta|^2 - \tau^2)(|\zeta|^2 - \tau^2)}{|\zeta|\left[(|\zeta|^2 - \tau^2 + 4m^2)\left((|\zeta|^2 - \tau^2)^2 + 4m^2|\zeta|^2\right)\right]^{1/2}}$$

and the observation

$$\int d\tau d\alpha d\zeta J|\hat{f}|^2|\hat{g}|^2 = \|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2} \quad .$$

B) Computation of $\hat{\Phi}_i(\tau,\zeta)$:

The computation of $E(f)E(g)(t,x)$ is similar. Let us mention what is different. In this case we set $\tau := \omega(\xi)+\omega(\eta)$, $\zeta := \xi+\eta$ and $X := \xi-\eta$, then $d\xi d\eta = cd\zeta dX$ and $X=(X_1,X_2)$ with X_1 in the direction of ζ satisfies the equation

$$\frac{\tau^2-|\zeta|^2}{\tau^2-|\zeta|^2-4m^2}\frac{X_1^2}{\tau^2}+\frac{X_2^2}{\tau^2-|\zeta|^2-4m^2}=1 \quad . \tag{1.10a}$$

Notice that $\tau^2-|\zeta|^2-4m^2 \geq 0$. Choosing natural coordinates on the ellipse defined by (1.10), namely

$$X_1 := a\cos\alpha \quad , \quad a := \frac{\tau\sqrt{\tau^2-|\zeta|^2-4m^2}}{\sqrt{\tau^2-|\zeta|^2}} \tag{1.10b}$$

$$X_2 := b\sin\alpha \quad , \quad b := \sqrt{\tau^2-|\zeta|^2-4m^2} \tag{1.10c}$$

we have for the Jacobian of the transformation $dX_1dX_2 = Jd\tau d\alpha$

$$J(\tau,\zeta,\alpha) = \frac{\tau^2(\tau^2-|\zeta|^2)-|\zeta|^2\left(\tau^2-|\zeta|^2-4m^2\right)\cos^2\alpha}{\left(\tau^2-|\zeta|^2\right)^{3/2}} \quad , \tag{1.11a}$$

while

$$4\omega(\xi)\omega(\eta) = \frac{\tau^2(\tau^2-|\zeta|^2)-|\zeta|^2\left(\tau^2-|\zeta|^2-4m^2\right)\cos^2\alpha}{\tau^2-|\zeta|^2} \quad . \tag{1.11b}$$

Finally we have the formula

$$\hat{\Phi}_i(\tau,\zeta) = \frac{1}{\sqrt{\tau^2-|\zeta|^2}}\int_{S^1}\hat{f}\left(\frac{\zeta+X(\alpha)}{2}\right)\hat{g}\left(\frac{\zeta-X(\alpha)}{2}\right)d\alpha \quad , \tag{1.12a}$$

where again $X(\alpha)$ can be expressed using complex notation

$$\zeta := |\zeta|e^{i\phi} \quad \text{and} \quad X(\alpha) = e^{i\phi}\left[a(\tau,\zeta)\cos\alpha + ib(\tau,\zeta)\sin\alpha\right] \tag{1.12b}$$

and a, b are defined in (1.10b,c) . We will use formula (1.12) later. Again the Lemma follows from the estimate

$$\frac{1}{\sqrt{\tau^2 - |\zeta|^2}} \int\limits_{S^1} \frac{d\alpha}{J} \sim \frac{H(\tau^2 - |\zeta|^2 - 4m^2)\sqrt{\tau^2 - |\zeta|^2}}{\tau \left[(\tau^2 - |\zeta|^2)^2 + 4m^2|\zeta|^2\right]^{1/2}}$$

and the computation

$$\int d\tau d\alpha d\zeta J |\hat{f}|^2 |\hat{g}|^2 = \|\hat{f}\|_{L^2} \|\hat{g}\|_{L^2} \quad .$$

This completes the proof of the Lemma.□

Remark : Notice that the previous computations can be combined in the case of the function $\phi(t, x)$ the solution of (0.1) as follows

$$\phi^2(t, x) = \frac{1}{\square^{1/2}} F(t, x)$$

where $F \in H^{1/2}(R^3)$ is given via the Fourier transform as the even function defined by

$$\hat{F}(\tau, \zeta) = i \int\limits_{S^1} \hat{f}\left(\frac{\zeta + X(\alpha)}{2}\right) \hat{g}\left(\frac{\zeta - X(\alpha)}{2}\right) d\alpha \quad \text{if} \quad \tau^2 - |\zeta|^2 > 0$$

and

$$\hat{F}(\tau, \zeta) = \int\limits_{R} \hat{f}\left(\frac{\zeta + X(\alpha)}{2}\right) \hat{g}\left(\frac{X(\alpha) - \zeta}{2}\right) d\alpha \quad \text{if} \quad |\zeta|^2 - \tau^2 > 0 \quad ,$$

while $\square^{1/2}$ is defined by

$$\hat{\square}^{1/2} := \begin{cases} \sqrt{|\zeta|^2 - \tau^2} & \text{if } |\zeta|^2 - \tau^2 > 0 \\ i\sqrt{\tau^2 - |\zeta|^2} & \text{if } \tau^2 - |\zeta|^2 > 0 \end{cases}$$

so that $\square^{1/2}\square^{1/2} = \square$. One can try to apply the energy estimate for the equation

$$\square \phi^2 = \square^{1/2} F$$

but the problem is that $F \in H^{1/2}(R^3)$ cannot be restricted on a hypersurface.

Notice that $\hat{\Phi}_i$ is supported in the region $\{\tau^2 - |\zeta|^2 - 4m^2 \geq 0\}$ while $\hat{\Phi}_o$ is supported in the region $\{|\zeta|^2 - \tau^2 \geq 0\}$. Combining both cases we can write

$$|\hat{\Phi}(\tau,\zeta)| \leq \hat{Q}(\tau,\zeta)\hat{D}(\tau,\zeta) \tag{1.13a}$$

where

$$\|\hat{D}\|_{L^2} \leq C\|f\|_{L^2}\|g\|_{L^2} \tag{1.13b}$$

and $\hat{Q}$ can be dominated using (1.4a,b), namely $\hat{Q}$ can be chosen to be

$$\hat{Q}(\tau,\zeta) = \frac{1}{[(\tau - |\zeta|)^2 + 4m^2]^{1/4}} + \frac{1}{[(\tau + |\zeta|)^2 + 4m^2]^{1/4}} \quad . \tag{1.13c}$$

Call

$$\tilde{Q}(t,\zeta) := \int_R \exp(it\tau)\hat{Q}(\tau,\zeta)d\tau \tag{1.13d}$$

the partial Fourier transform with respect to the dual time variable. Then a simple change of coordinates gives

$$\tilde{Q}(t,\zeta) = 2\cos(t|\zeta|)\int_R \frac{\exp(it\tau)d\tau}{[\tau^2 + 4m^2]^{1/4}} \quad . \tag{1.13e}$$

We want to isolate the singular part in the above expression, for this reason call

$$k(t) := \int_R \frac{\exp(it\tau)d\tau}{[\tau^2 + 4m^2]^{1/4}} \quad . \tag{1.14}$$

Now the function $k(t)$ has the following properties stated in the lemma below.

Lemma 1.2 : *The function $k(t)$ defined in (1.14) satisfies the following properties*

$$|k(t)| \leq \frac{C}{|t|^{1/2}} \qquad \text{if} \qquad |t| < 1 \tag{1.15a}$$

$$|k(t)| \leq \frac{C}{|t|\sqrt{m}} \qquad \text{if} \qquad |t| \geq 1 \tag{1.15b}$$

Proof :

To prove (1.15b) notice that by integration by parts we have

$$\int \frac{\exp(it\tau)d\tau}{[\tau^2+4m^2]^{1/4}} = \frac{1}{2it}\int \frac{\exp(it\tau)\tau d\tau}{[\tau^2+4m^2]^{5/4}}$$

and

$$\int_0^\infty \frac{\tau d\tau}{[\tau^2+4m^2]^{5/4}} \sim \frac{C}{\sqrt{m}}$$

In order to prove (1.15a) it is enough to notice that if $|\tau|$ is large then

$$\frac{1}{[\tau^2+4m^2]^{1/4}} \sim \frac{1}{|\tau|^{1/2}}$$

and this completes the proof. □

We need one more technical lemma.

Lemma 1.3 : *Assume that $k(t) \geq 0$ satisfies the conditions (1.15a,b) from Lemma 1.2. Consider the operator defined by*

$$I(f)(t) := \int_R k(t-s)f(s)ds \tag{1.16a}$$

where $f(s)$ has compact support in the interval $[-T,T]$. Then

$$\|I(f)\|_{L^2(R)} \leq C\|f\|_{h^1} \tag{1.16b}$$

Proof :

Compute

$$\int_R I^2(f)(t)dt = \int_R dt \int_{[-T,T]\times[-T,T]} k(t-s_1)k(t-s_2)f(s_1)f(s_2)ds_1ds_2$$

$$= \int_{[-T,T]\times[-T,T]} h(s_1,s_2)f(s_1)f(s_2)ds_1ds_2$$

where

$$h(s_1,s_2) := \int_R k(t-s_1)k(t-s_2)dt$$

It is easy to see that because of (1.15a,b) we have

$$h(s_1,s_2) \sim |\log|s_1-s_2|| \qquad \text{if} \qquad |s_1-s_2|<1$$

while $h(s_1,s_2)$ is bounded if $|s_1-s_2|\geq 1$. Hence for fixed s_1 we have

$$\|h(s_1,.)\|_{bmo} \leq C$$

independent of s_1. Now

$$\int I^2(f)(t)dt \leq \|f\|_{L^1} \sup_{s_1} \|h(s_1,.)\|_{bmo}\|f\|_{h^1} \quad .$$

This proves the Lemma. □

Now we are in the position to prove

Theorem 1.1 : *Call*

$$\epsilon(t) := \left[\int_{R^2} \left|D_{t,x}\left(E(f)\bar{E}(g)\right)\right|^2 dx\right]^{1/2} \tag{1.17a}$$

or

$$\epsilon(t) := \left[\int_{R^2} \left|D_{t,x}\left(E(f)E(g)\right)\right|^2 dx\right]^{1/2} \tag{1.17b}$$

then there is a positive function $b(t)$ defined almost everywhere such that

$$\epsilon(t) \leq b(t) \tag{1.18a}$$

and

$$\|b\|_{bmo} \leq C\|f\|_{L^2}\|g\|_{L^2} \quad . \tag{1.18b}$$

Proof :

Because of Lemma 1.1 the Fourier transform of both quantities

$$D_{t,x}\left(E(f)E(g)\right) \quad \text{and} \quad D_{t,x}\left(E(f)\bar{E}(g)\right)$$

can be written as $\hat{G}$ which satisfies

$$|\hat{G}(\tau,\zeta)| \le \hat{Q}(\tau,\zeta)\hat{D}(\tau,\zeta)$$

where $\hat{D}$ and $\tilde{Q}$ satisfy

$$\|\hat{D}\|_{L^2} \le C\|f\|_{L^2}\|g\|_{L^2}$$

and

$$\tilde{Q}(t,\zeta) = 2\cos(t|\zeta|)k(t)$$

with $k(t)$ defined in (1.14). Using tilde to denote the partial Fourier transform with respect to the dual time variable we have

$$\tilde{G}(t,\zeta) = \int\limits_R 2\cos\left((t-s)|\zeta|\right)k(t-s)\tilde{D}(s,\zeta)ds$$

and from this it follows that

$$\left\|\tilde{G}(t,\cdot)\right\|_{L^2} \le \int\limits_R k(t-s)\left\|\tilde{D}(s,\cdot)\right\|_{L^2} ds := b(t)$$

We want to use a duality argument. Pick an arbitrary function $l(t)$ with support in the time interval $t \in [-T,T]$ and compute

$$\begin{aligned} |\,(l,b)\,| &\le \int k(t-s)\|\tilde{D}(s,\cdot)\|_{L^2} l(t,)dtds \\ &\le \|D\|_{L^2}\left\|\int k(t-s)l(t,)dt\right\|_{L^2} \\ &\le \|D\|_{L^2} C\|l(\cdot)\|_{h^1} \end{aligned}$$

since $k(t)$ satisfies (1.15a,b) in Lemma 1.2 . Hence b defines a bounded functional on h^1 and

$$\|b(\cdot)\|_{bmo} \le C\|D\|_{L^2} \le C\|f\|_{L^2}\|g\|_{L^2}$$

which concludes the proof. □

Now we want to give the proof of the estimates in Theorems (0.2,3). Let us state the basic estimate from which all the others follow.

Theorem 1.2 : *Consider the integral operator of the type*

$$E(f)(t,x) := \int_{R^2} \exp\{i(t\omega(\xi) + x\cdot\xi)\}\hat{f}(\xi)d\mu(\xi) \qquad (1.19a)$$

$$\omega(\xi) = \sqrt{|\xi|^2 + m^2} \quad \text{with} \quad d\mu(\xi) := \frac{d\xi}{\omega(\xi)} \qquad (1.19b)$$

Denote by $\hat{\Phi}_{i,o}$ the Fourier transform of $E(f)E(g)$ and $E(f)\bar{E}(g)$ respectively. The following estimates hold :

$$\left\| |\tau^2 - |\zeta|^2|^{1/4}\hat{\Phi}_{i,o}(\tau,\zeta)\right\|_{L^2(R^3)} \le C\|\hat{f}(\xi)\|_{L^2(d\mu)}\|\hat{g}(\xi)\|_{L^2(d\mu)} \qquad (1.20a)$$

and

$$\left\| (||\tau| - |\zeta||)^{1/2}\hat{\Phi}_{i,o}(\tau,\zeta)\right\|_{L^2} \le C\left\|\hat{f}/\omega\right\|_{L^2(R^2)}\|\hat{g}\|_{L^2(R^2)} \quad . \qquad (1.20b)$$

Proof :

Consider first the quantity $\hat{\Phi}_i$ which is the integral given in (1.12a), recall that $\hat{\Phi}_i$ is supported in the region $\tau \ge 0$ and $\tau^2 - |\zeta|^2 \ge 4m^2$. Both estimates follow from the same computation so we will prove only (1.20b). Set

$$\xi_{1,2} := \frac{\zeta \pm X(\alpha)}{2} \qquad .$$

Cauchy-Schwartz inequality gives

$$(\tau - |\zeta|)\left|\hat{\Phi}_i\right|^2 \le$$

$$\left(\frac{\tau - |\zeta|}{\sqrt{\tau^2 - |\zeta|^2}}\int_{S^1}\frac{\omega(\xi_1)}{\omega(\xi_2)}d\alpha\right)\int_{S^1}|\hat{f}(\xi_1)|^2|\hat{g}(\xi_2)|^2\frac{\omega(\xi_2)}{\omega(\xi_1)}\frac{d\alpha}{\sqrt{\tau^2 - |\zeta|^2}} \qquad .$$

Observe first that, see (1.11a,b)

$$\frac{d\tau d\alpha d\zeta}{\sqrt{\tau^2 - |\zeta|^2}} = \frac{d\xi_1 d\xi_2}{\omega(\xi_1)\omega(\xi_2)} \qquad ,$$

and the estimate will follow after the inequality with respect to τ, ζ, provided that the quantity

$$\frac{\tau - |\zeta|}{\sqrt{\tau^2 - |\zeta|^2}} \int\limits_{S^1} \frac{\omega(\xi_1)}{\omega(\xi_2)} d\alpha$$

is bounded for every τ , ζ. It is straightforward to calculate

$$\omega(\xi_{1,2}) = \tau \pm q|\zeta| \cos\alpha \qquad \text{with} \qquad q := \sqrt{1 - \frac{4m^2}{\tau^2 - |\zeta|^2}} \qquad .$$

Call $\lambda := \tau/|\zeta|q$. We have

$$\int\limits_{S^1} \frac{\omega(\xi_1)}{\omega(\xi_2)} d\alpha = \int\limits_{S^1} \frac{\lambda + \cos\alpha}{\lambda - \cos\alpha} d\alpha \sim \frac{1}{\sqrt{\lambda^2 - 1}}$$

and the quantity

$$\frac{\tau - |\zeta|}{\sqrt{(\tau^2 - |\zeta|^2)(\lambda^2 - 1)}}$$

is bounded for every τ and ζ.

In order to prove the estimates for the term $\hat{\Phi}_o$, see (1.9a), we need a new idea. Consider a dyadic decomposition of the functions $\hat{f}$ and $\hat{g}$. This can be achieved as follows, we start with a smooth function $\hat{a}$ supported in the set $\{1/2 \leq |\xi| \leq 1\}$ and form the dyadic dilations $\hat{a}_j(\xi) := \hat{a}(\xi/2^j)$. It is easy to see that $\hat{a}$ can be chosen so that

$$\sum_{j=-\infty}^{+\infty} \hat{a}_j(\xi) = 1 \qquad \text{if} \qquad \xi \neq 0 \quad .$$

Now set

$$\hat{a}_0(\xi) := 1 - \sum_{j=1}^{+\infty} \hat{a}_j(\xi)$$

and define

$$\hat{f}_j(\xi) := \hat{a}_j(\xi)\hat{f}(\xi) \qquad \text{where} \qquad j = 0, 1, \cdots + \infty \quad .$$

Denote the dyadic components f_j and g_k and consider the coresponding expression $\hat{\Phi}_{o,jk}$. The proof (1.20a) will follow from the estimate

$$\left\|(|\zeta|^2-\tau^2)^{1/4}\hat{\Phi}_{o,jk}\right\|_{L^2} \leq \frac{C}{2^{|j-k|/4}}\|\hat{f}_j\|_{L^2(d\mu)}\|\hat{g}_k\|_{L^2(d\mu)} \quad . \tag{1.21}$$

Observe that

$$\omega(\xi_{1,2}) = |\zeta|q\cosh\alpha \pm \tau \qquad \text{with} \qquad q := \sqrt{1+\frac{4m^2}{|\zeta|^2-\tau^2}}$$

and, see (1.8a,b),

$$\frac{d\tau d\alpha d\zeta}{\sqrt{|\zeta|^2-\tau^2}} = \frac{d\xi_1 d\xi_2}{\omega(\xi_1)\omega(\xi_2)} \quad .$$

Case A : Assume that $j \sim k$ and both j and k large, otherwise there is nothing to prove. In this case

$$\frac{\omega(\xi_1)}{\omega(\xi_2)} = \frac{\cosh\alpha+\lambda}{\cosh\alpha-\lambda} \sim 1 \quad \text{with} \quad \lambda := \frac{\tau}{|\zeta|q} < 1 \quad .$$

It is easy to see that for any λ the variable α has to belong to a bounded interval. The most interesting case is when λ is close to one, in which case α has to be large and belong to an interval of the form

$$e^\alpha \in \left[\frac{2^j}{|\zeta|q}, \frac{2^{j+1}}{|\zeta|q}\right] \quad .$$

Hence using Cauchy-Schwartz again

$$\left|\hat{\Phi}_{o,jk}\right|^2 \leq \left(\int d\alpha\right)\int |\hat{f}(\xi_1)|^2|\hat{g}(\xi_2)|^2\frac{d\alpha}{|\zeta|^2-\tau^2} \quad .$$

Multiplying the above inequality with $\sqrt{|\zeta|^2-\tau^2}$ and integrating with respect to τ and ζ gives (1.21).

Case B : Assume $k << j$. In this case $|\zeta| \sim 2^j$ and $\tau \sim 2^j$. Without loss of generality we can assume that $\tau > 0$. Now

$$\frac{\omega(\xi_1)}{\omega(\xi_2)} = \frac{\cosh\alpha+\lambda}{\cosh\alpha-\lambda} \sim \frac{2^j}{2^k} >> 1 \quad ,$$

where $\lambda := \tau/|\zeta| q \sim 1$. For this to be satisfied α must be close to zero and in particular

$$\alpha^2 \leq \frac{2^k}{2^j} \qquad \text{hence} \qquad \int d\alpha \sim \frac{2^{k/2}}{2^{j/2}} \qquad .$$

Now we have after using Cauchy-Schwartz again

$$\left|\hat{\Phi}_{o,jk}\right|^2 \leq \frac{2^{k/2}}{2^{j/2}} \int |\hat{f}(\xi_1)|^2 |\hat{g}(\xi_2)|^2 \frac{d\alpha}{|\zeta|^2 - \tau^2} \quad ,$$

from which the estimate follows after integration in τ and ζ.

Finally we want to prove the estimate

$$\left\| (|\zeta| - |\tau|)^{1/2}\, \hat{\Phi}_o \right\|_{L^2} \leq C \|\hat{f}/\omega\|_{L^2} \|\hat{g}\|_{L^2} \quad .$$

Write

$$\hat{\Phi}_o = \sum_{j,k} \hat{\Phi}_{o,jk} = \sum_{|j-k|>2} \hat{\Phi}_{o,jk} + \sum_{|j-k|\leq 2} \hat{\Phi}_{o,jk} = \hat{\Phi}_o^1 + \sum_{|j-k|\leq 2} \hat{\Phi}_{o,jk} \quad . \tag{1.22}$$

The first term in the above expression can be estimated as follows. Notice first that

$$\hat{\Phi}_o^1 = \frac{1}{\sqrt{|\zeta|^2 - \tau^2}} \int \hat{f}\left(\frac{\zeta + X(\alpha)}{2}\right) \bar{\hat{g}}\left(\frac{X(\alpha) - \zeta}{2}\right) \chi_A(\tau, \zeta, \alpha) d\alpha \quad ,$$

where $\chi_A(\tau, \alpha, \zeta)$ is the characteristic function of a set contained in

$$A \subset \left\{ (\tau, \alpha, \zeta) \quad : \quad \frac{\omega(\xi_1)}{\omega(\xi_2)} \leq \frac{1}{2} \quad \text{or} \quad \frac{\omega(\xi_1)}{\omega(\xi_2)} \geq 2 \right\} \quad .$$

The relation

$$\frac{\omega(\xi_1)}{\omega(\xi_2)} = \frac{\cosh\alpha + \lambda}{\cosh\alpha - \lambda} \qquad \text{with} \qquad \lambda := \frac{\tau}{|\zeta| q} < 1$$

implies that for every fixed (τ, ζ) the range of possible values of α belong to a bounded set i.e $\{\alpha \in R : |\alpha| \leq C\}$. Using Cauchy-Schwartz one more time we have

$$(|\zeta| - |\tau|) \left|\hat{\Phi}_o^1\right|^2 \leq$$
$$\left(\frac{|\zeta| - |\tau|}{\sqrt{|\zeta|^2 - \tau^2}} \int_{|\alpha|<C} \frac{\omega(\xi_1)}{\omega(\xi_2)} d\alpha \right) \int |\hat{f}(\xi_1)|^2 |\hat{g}(\xi_2)|^2 \frac{\omega(\xi_2)}{\omega(\xi_1)} \frac{d\alpha}{\sqrt{|\zeta|^2 - \tau^2}} \quad .$$

¿From the observation

$$\int\limits_{|\alpha|<C} \frac{\omega(\xi_1)}{\omega(\xi_2)} d\alpha = \int\limits_{|\alpha|<C} \frac{\cosh\alpha + \lambda}{\cosh\alpha - \lambda} d\alpha \sim \frac{1}{\sqrt{1-\lambda^2}} \quad \text{where} \quad \lambda := \frac{\tau}{|\zeta| q} < 1$$

we have after integration with respect to τ, ζ the estimate

$$\left\| (|\zeta| - |\tau|)^{1/2} \, \hat{\Phi}_o^1 \right\|_{L^2} \leq C \|\hat{f}/\omega\|_{L^2} \|\hat{g}\|_{L^2} \quad .$$

To handle the last term in the expansion (1.22) observe that

$$(|\zeta| - |\tau|) \left| \hat{\Phi}_{o,jk} \right|^2 \leq \frac{|\zeta| - |\tau|}{\sqrt{|\zeta|^2 - \tau^2}} \left(\int d\alpha \right) \int |\hat{f}_j|^2 |\hat{g}_k|^2 \frac{d\alpha}{\sqrt{|\zeta|^2 - \tau^2}} \quad .$$

Since $|j-k| \leq 2$ the variable α must belong to a bounded interval. Integrating with respect to τ and ζ we obtain

$$\left\| \sqrt{|\zeta| - |\tau|} \hat{\Phi}_{o,jk} \right\|_{L^2} \leq C \|\hat{f}_j/\omega\|_{L^2} \|\hat{g}_k\|_{L^2} \quad .$$

Summing over j, k and taking into account the fact that $j \sim k$ gives the desired estimate. □

The estimate of Strichartz follows from the multiplier estimate

$$\|f\|_{L^3(R^3)} \leq \| \left| \tau^2 - |\zeta|^2 \right|^{1/4} \hat{f}(\tau, \zeta) \|_{L^2(R^3)}$$

see [3].

Remark : A more careful analysis taking into account the fact that $\hat{\Phi}_i$ is supported in the region where $\tau^2 - |\zeta|^2 - 4m^2 \geq 0$ and the observation that $\hat{\Phi}_o$ is the convolution over a hyperbola that vanishes when $|\zeta| = |\tau|$ can prove the following slight improvement.

$$\left\| (||\zeta| - |\tau|| + m)^{1/2} \, \hat{\Phi}_{i,o} \right\| \leq C \|\hat{f}/\omega\|_{L^2} \|\hat{g}\|_{L^2} \quad . \tag{1.23}$$

Now we are in the position to prove Theorem (0.2).

Proof of Theorem 0.2 :

Observe that from Theorem 1.2 we have

$$(|\tau| + |\zeta|)^{1/4} \hat{\Phi}_{i,o} \sim \frac{\hat{D}_{i,o}}{||\tau| - |\zeta||^{1/4}} \quad ,$$

where

$$\|\hat{D}_{i,o}\|_{L^2} \leq C \|\hat{f}\|_{L^2(d\mu)} \|\hat{g}\|_{L^2(d\mu)} \quad .$$

Denote by $\tilde{}$ the partial Fourier transform with respect to the time variable and observe that

$$\int_R \frac{e^{i\tau t} d\tau}{||\tau| - |\zeta||^{1/4}} \sim \frac{e^{it|\zeta|}}{|t|^{3/4}} \qquad \text{for} \qquad |t| \qquad \text{small} \quad .$$

Consider an arbitrary function $l(t,x)$. A duality argument gives

$$\begin{aligned} \left| \left(\hat{l}, D^{1/4} \hat{\Phi}_{i,o} \right) \right| &\leq \int\limits_{R \times R} \frac{1}{|t-s|^{3/4}} \|\tilde{l}(t,\cdot)\|_{L^2} \left\| \widetilde{D}_{i,o}(s,\cdot) \right\|_{L^2} dt ds \\ &\leq \|D_{i,o}\|_{L^2} \left\| \int\limits_R \frac{\|l(t,\cdot)\|_{L^2}}{|t-s|^{3/4}} dt \right\|_{L^2(R)} \\ &\leq C \|D_{i,o}\|_{L^2} \left\| \|l(t,\cdot)\|_{L^2(R^2)} \right\|_{L^{8/3}(R)} \quad , \end{aligned}$$

where we used the Littlewood Hardy Polya inequality in the last step. This concludes the proof. □

Proof of Theorem 0.3 :

We will give only an outline. The proof of Theorem 0.3 follows exactly the same steps as in the proof of Theorem 1.1, starting from the observation that because of (1.23) we have

$$\hat{\Phi}_{i,o} \sim \frac{1}{(||\tau| - |\zeta|| + m)^{1/2}} \hat{D}_{i,o} \quad ,$$

where

$$\left\| \hat{D}_{i,o} \right\|_{L^2(R^3)} \leq C \left\| \hat{f}/\omega \right\|_{L^2(R^2)} \|\hat{g}\|_{L^2(R^2)} \quad .$$

.□

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