Journal of Hyperbolic Differential Equations
Vol. 4, No. 2 (2007) 1–25

ON THE GLOBAL EXISTENCE OF ROUGH SOLUTIONS OF THE CUBIC DEFOCUSING SCHRÖDINGER EQUATION IN $\mathbf{R}^{2+1}$

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Received 23 May 2006
Accepted 10 Oct. 2006

Communicated by P. G. LeFloch

Abstract. We consider the cubic defocusing Schrödinger equation in two space dimensions and prove that if the initial data are in $H^{1/2}$, then there exists a global solution in time. The proof combines the argument from [5] with some new correlation estimates for the Schrödinger equation.

Keywords: I-method; Strichartz estimates; correlation estimates; conservation laws.

Mathematics Subject Classification 2000: 35L40, 76N10

1. Introduction

The purpose of the present paper is to demonstrate global existence of rough solutions for the cubic defocusing Schrödinger equation in $2+1$ dimensions. To make the statement more precise, we would like to consider the following evolution equation

$$i\partial_t\psi - \Delta\psi + |\psi|^2\psi = 0, \tag{1.1a}$$

$$\psi(0,x) = \psi_0(x); \quad (t,\mathbf{x}) \in \mathbf{R}\times\mathbf{R}^2, \tag{1.1b}$$

i.e. $\mathbf{x} \in R^2$ are the space variables and t denotes the time variable. For the evolution Eq. (1.1), we would like to assume that $\psi_0 \in H^{1/2}(\mathbf{R}^2)$ and show that there exists a global in time solution with $\psi(t) \in H^{1/2}(\mathbf{R}^2)$. Certain conserved quantities are crucial to the behaviour of the solutions. The conservation of energy means that the integral

$$E(t) \stackrel{\text{def}}{=} \int_{R^2} \left(\left|\nabla\psi\right|^2 + \frac{1}{2}|\psi|^4\right)\, d\mathbf{x} \tag{1.2a}$$

is independent of time. On the other hand, the evolution equation preserves the total mass which means that the integral

$$M(t) \stackrel{\text{def}}{=} \int_{R^2} |\psi|^2 d\mathbf{x} \tag{1.2b}$$

is also independent of time. It is rather easy to demonstrate, using the two conserved quantities above, that a solution of (1.1) exists for all time provided that the initial data are sufficiently smooth, namely in H^1. The reason for assuming $\psi_0 \in H^1$ is that one would like to take advantage of the conservation of energy, see the intergal in (1.2a), which is positive definite and which implies that the H^1 norm of the solution is bounded as long as the solution exists. In order to explain the type of result that we would like to demonstrate in the present work, we have to explain first why the Eq. (1.1) admit local weak solutions if we only assume that the initial data are in L^2. To see why this is so, we have to use the Strichartz estimates, more precisely consider the linear Schrödinger equation with a forcing term f,

$$i\psi_t - \Delta\psi = f, \tag{1.3a}$$

$$\psi(0, \mathbf{x}) = \psi_0(\mathbf{x}). \tag{1.3b}$$

The solution of (1.3) can be written in integral form as a combination of two terms, namely $\psi(t) = \psi_L(t) + \phi(t)$ where

$$\psi_L(t) \stackrel{\text{def}}{=} e^{-i\Delta t}\psi_0; \quad \phi(t) \stackrel{\text{def}}{=} \int_0^t e^{-i\Delta(t-s)} f(s)ds. \tag{1.4}$$

The local in time construction of solutions is based on the well known Strichartz estimates

$$\|\psi_L\|_{L^4(R^{2+1})} \leq C_1\|\psi_0\|_{L^2(R^2)}; \quad \|\phi\|_{L^4(R^{2+1})} \leq C_2\|f\|_{L^{4/3}(R^{2+1})}. \tag{1.5}$$

Pick a time interval, say $[0, T]$, with the final time T to be chosen and set up an iteration scheme as follows

$$i\psi_t^{(k+1)} - \Delta\psi^{(k+1)} = -\chi_{[0,T]}(t)|\psi^{(k)}|^2\psi^{(k)}; \quad \psi^{(k+1)}(0, \mathbf{x}) = \psi_0(\mathbf{x}) \tag{1.6}$$

with $\psi^{(0)}$ satisfying $i\partial_t\psi^{(0)} - \Delta\psi^{(0)} = 0$, $\psi^{(0)}(0) = \psi_0$ and $\chi_{[0,T]}$ the characteristic function of the interval $[0, T]$. The Strichartz estimate in (1.5) $\|\psi_L\|_{L^4(R^{2+1})} \leq C_1\|\psi_0\|_{L^2(R^2)}$ implies that there exists some time interval, say $[0, T]$, such that

$$\|\psi_L\|_{L^4(R^2\times[0,2T])} \leq \epsilon, \tag{1.7}$$

where ϵ is a fixed small number which will depend only on C_2, see (1.5). Let us define a local in time norm,

$$X_k =^{\text{def}} \|\psi^{(k)}\|_{L^4(R^2\times[0,T])}. \tag{1.8}$$

Applying the estimates (1.5), we obtain an inequality of the form $X_{k+1} \leq \epsilon + CX_k^3$, from which it follows that X_k is a bounded sequence. By considering the evolution equation for differences $\psi^{(k+1)} - \psi^{(k)}$, one can show that the iteration is in fact a contraction with respect to the L^4 norm, thus constructing a local solution of (1.1) in the time interval $[0, T]$. A crucial observation is the fact that T is defined via (1.7) and we do not have any control of its size. Repeating the construction starting with data at time T, we can construct a solution on successive time intervals, but there is no guarantee that the intervals will not shrink rapidly thus constructing a solution only on a finite time interval. On the other hand, if we use the conservation of energy this cannot happen, since the energy estimate controls the H^1 norm of the solution, but we need to assume that $\psi_0 \in H^1$. It is an interesting question to ask whether equation (1.1) has global in time solutions if one assumes that $\psi_0 \in H^s$ for $0 \leq s < 1$. This question was first posed by Bourgain, see [1, 2], in which he demonstrated global existence for $s > 3/5$. The range of s was subsequently improved in [5, 6] to $s > 4/7$ using the crucial idea of an almost conservation law for the energy. In the present work we would like to demonstrate global existence for $s = 1/2$. There are two key ingredients needed in the proof, the first is the almost conservation law and the second is a correlation estimate. The paper is divided as follows, in Sec. 2, we present the main argument leading to global existence, we essentially follow the argument in [6]. In Secs. 3 and 4, we derive a bilinear estimate due to Bourgain and the almost conservation law developed in [5, 6]. Section 5 presents a method for deriving correlation estimates. Recently Coliander *et al.* showed existence for $s > 1/2$ by improving the almost conservation law.

2. Main Argument

Presently we would like to give the main argument of the proof of global existence for the following problem

$$i\partial_t \psi - \Delta\psi + |\psi|^2\psi = 0, \tag{2.1a}$$

$$\psi(0, x) := \psi_0(x) \in H^{1/2}(R^2). \tag{2.1b}$$

We will follow the technique used in [5, 6] which is based on the multiplier I and the idea of an almost conservation law. Before we start let us specify some notation conventions. We would like to define the Fourier multiplier $m_{s,N}(\xi)$ to be a smooth function in frequency variables which is less or equal than one and has the following properties

$$m_{s,N}(\xi) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } |\xi| \leq N, \\ (N/|\xi|)^{1-s}, & \text{if } |\xi| \geq 2N, \end{cases} \tag{2.2}$$

for $0 < s < 1$, however in the present work we will choose $s = 1/2$. We will use the multiplier above to define $I\psi$, which whenever convenient we will also denote by ψ_I, as follows

$$\psi_I(t, x) = I\psi(t, x) =^{\text{def}} \mathcal{F}^{-1}\{m_{s,N}(\xi)\widehat{\psi}(t, \xi)\}. \tag{2.3}$$

We will also use the following spaces introduced by Bourgain, see [1, 2].

$$\|\psi\|_{X_{s,\alpha}} \stackrel{\text{def}}{=} \|\langle\xi\rangle^s \langle\widehat{S}\rangle^\alpha \widehat{\psi}\|_{L^2}, \tag{2.4a}$$

where we adopt the notation convention

$$\langle\xi\rangle \stackrel{\text{def}}{=} |\xi| + 1; \quad \widehat{S} \stackrel{\text{def}}{=} \tau - |\xi|^2; \quad \langle\widehat{S}\rangle \stackrel{\text{def}}{=} |\widehat{S}| + 1. \tag{2.4b}$$

When we consider localization in time, we will use the notation $X^I_{s,\alpha}$ or simply write $X^{loc}_{s,\alpha}$, where I stands for some time interval $[T_1, T_2]$. This means that we consider

$$\|\psi\|_{X^I_{s,\alpha}} \stackrel{\text{def}}{=} \inf_{\widetilde{\psi}} \left\{\|\widetilde{\psi}\|_{X_{s,\alpha}} : \ \widetilde{\psi}(t) = \psi(t), \ t \in I\right\}. \tag{2.5}$$

The s-derivative of a function is defined in the standard way

$$D^s\psi = \mathcal{F}^{-1}\left\{\langle|\xi|\rangle^s \widehat{f}\right\}. \tag{2.6}$$

The energy of the function ψ_I is defined as follows, see the definition of energy in (1.2a),

$$E(I\psi)(t) \stackrel{\text{def}}{=} \int_{R^2_t} \left\{\left|\nabla\psi_I\right|^2 + \frac{1}{2}\left|\psi_I\right|^4\right\} d\mathbf{x}. \tag{2.7}$$

Finally, we will employ the Strichartz estimates

$$\left\|\int_0^t e^{-i(t-s)\Delta} f(s)ds\right\|_{L^p_x, L^r_t} \le C_1 \left\|f\right\|_{L^{p'}_x, L^{r'}_t}, \tag{2.8a}$$

$$\left\|e^{-it\Delta}\psi_0\right\|_{L^p_x, L^r_t} \le C_0 \left\|\psi_0\right\|_{L^2(R^2)}, \tag{2.8b}$$

where the exponents p and r satisfy the admissibility condition, see [5, 6],

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{2}; \quad r > 2. \tag{2.8c}$$

Let us list here the ingredients that are essential for the demonstration of global existence. The first observation is the fact that if $\psi(t, \mathbf{x})$ is a solution of (2.1), we can scale it and obtain a new solution, namely the scaled function,

$$\psi^{(\lambda)}(t, x) \stackrel{\text{def}}{=} \frac{1}{\lambda}\psi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \tag{2.9}$$

satisfies the same equation with initial data $\psi^{(\lambda)}_0 = (1/\lambda)\psi_0(x/\lambda)$. This scaling preserves the L^2 norm of $\psi(t)$ as well as the L^4 space-time norm. Now let us scale the solution $\psi \mapsto \psi^{(\lambda)}$ so that $E(I\psi^{(\lambda)}_0) \le 1/4$, more precisely under the scaling in (2.9), we have

$$\|\nabla I\psi^{(\lambda)}_0\|^2_{L^2} \approx \frac{N}{\lambda}\|\psi_0\|^2_{H^{1/2}} \quad \text{and} \quad \|I\psi^{(\lambda)}_0\|^4_{L^4} \approx \frac{1}{\lambda^2}\|\psi_0\|^4_{L^4}. \tag{2.10}$$

Notice that from the Sobolev inequality we have $\|\psi\|_{L^4} \leq C\|\psi\|_{H^{1/2}}$. With all the above in mind, we can choose λ in the following manner

$$\lambda = CN\|\psi_0\|^2_{H^{1/2}}, \tag{2.11}$$

where C some fixed large positive number so that the scaled function $\psi_0^{(\lambda)}$ satisfies,

$$E(I\psi_0^{(\lambda)}) < 1/4. \tag{2.12}$$

We will choose N at the end of the argument to be sufficiently large.

The almost conservation law is the next essential ingredient, see [5, 6] where this idea was introduced. The idea is that the energy of ψ_I is almost conserved in the following sense

$$\big|E(I\psi)(T) - E(I\psi)(0)\big| \leq \frac{1}{N^{(3-\delta)/2}}\left[C_1\|I\psi\|^4_{X^{[0,T]}_{1,(1+\delta)/2}} + C_2\|I\psi\|^6_{X^{[0,T]}_{1,(1+\delta)/2}}\right]. \tag{2.13}$$

The third crucial ingredient is an *a priori* estimate which reads as follows. A solution of (2.1) satisfies the *a priori* estimate

$$\|\psi\|^4_{L^4(\mathbf{R}^2\times[T_1,T_2])} \leq \|\psi_0\|^2_{L^2}\left(\sup_{[T_1,T_2]}\|\psi(t)\|^2_{H^{1/2}}\right)\sqrt{T_2 - T_1}. \tag{2.14}$$

This type of estimate for three space dimensions was derived first in [6, 7]. A general method for developing this type of apriori estimates will be given in Sec. 5. In order to present the proof of global existence, we will need a series of technical lemmata. The first lemma that we will need is the following, see [3].

Lemma 2.1. *Assume that* $q(\xi_1, \xi_2)$, *with* $\xi_{1,2} \in R^2$, *is a smooth function satisfying*

$$|\partial^\alpha q| \leq C(\alpha)(1 + |\xi|)^{-|\alpha|}; \quad \xi = (\xi_1, \xi_2) \tag{2.15}$$

for every multiindex α *and define a quadratic expression as follows*

$$Q[f,g](\mathbf{x}) = \mathcal{F}^{-1}\left\{\int q(\xi - \eta, \eta)\widehat{f}(\xi - \eta)\widehat{g}(\eta)d\eta\right\}. \tag{2.16}$$

The following estimate holds

$$\|Q[f,g]\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}, \quad \text{where} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}. \tag{2.17}$$

This lemma appears in [3] and we will omit its proof. We will use Lemma 2.1 above to claim that as far as the L^p norm is concerned we can think that $DI(\psi^3) \sim (\psi^2)(DI\psi)$. In order to apply Lemma 2.1 we need to assume that the multiplier I has smooth symbol which can be achieved by considering a smooth approximation of $m_{s,N}$, see (2.2). We will also need the following Strichartz type estimate.

Theorem 2.2. *A pair of exponents is called admissible if it satisfies*

$$\frac{1}{p}+\frac{1}{r}=\frac{1}{2}; \quad r>2 \tag{2.18a}$$

and with this choice of exponents the following Strichartz type estimates holds true

$$\|\psi\|_{L_x^p,L_t^r}\le C(\delta)\|\psi\|_{X_{0,(1+\delta)/2}}. \tag{2.18b}$$

These estimates are well known, see Bourgain [1, 2], also see next section. Next we will need a lemma that says that if the space-time L^4 norm of ψ is small on some interval $[0,T]$, we can control all higher order norms.

Lemma 2.3 (Local existence with $H^s(\mathbf{R}^2)$ data, where $s\ge 0$). *Let us define the quantity,*

$$\mu([0,t])\stackrel{\text{def}}{=}\int_{D[0,t]}|\psi|^4\,d\mathbf{x}dt. \tag{2.19a}$$

If $\mu([0,T])\le\mu_0$, where μ_0 is some universal constant then the following estimates are true,

$$\big\|D^s\psi\big\|_{L_x^pL_t^r(D[0,T])}\le C\big\|D^s\psi_0\big\|_{L^2}, \tag{2.19b}$$

$$\big\|DI\psi\big\|_{L^4(D[0,T])}\le C\big\|DI\psi_0\big\|_{L^2}. \tag{2.19c}$$

Proof. The proof imitates the argument for local existence given in the introduction, the function $D^s\psi$ satisfies the equation,

$$i\partial_t\big(D^s\psi\big)-\Delta\big(D^s\psi\big)=-D^s\big(\chi_{[0,T]}(t)|\psi|^2\psi\big) \tag{2.20}$$

in the interval $[0,T]$, where $\chi_{[0,T]}$ denotes the characteristic function of the time interval $[0,T]$. Let us take $p=r=4$ for simplicity. Let us write $D:=\mathbf{R}^2\times[0,T]$ for simplicity. Applying Strichartz estimates as stated in (2.8a) and (2.8b), we obtain

$$\big\|D^s\psi\big\|_{L^4(D)}\le C_1\big\|D^s\psi_0\big\|_{L^2}+C_2\big\|D^s\big(|\psi|^2\psi\big)\big\|_{L^{4/3}(D)}. \tag{2.21}$$

Because of Lemma 2.1, we can write $D^s(|\psi|^2\psi)\sim\psi^2(D^s\psi)$ and using Holder on the right-hand side of (2.21) with exponents 4 and 4/3, we obtain

$$\big\|D^s\psi\big\|_{L^4(D)}\le C_1\big\|D^s\psi_0\big\|_{L^2}+C_2\big[\mu^{1/2}([0,T])\big]\big\|D^s\psi\big\|_{L^4(D)}. \tag{2.22}$$

We need to assume $C_2^2\mu([0,T])\le 1/4$ in order to complete the proof of the lemma. The proof of (2.19b) is similar so we will omit it. □

Lemma 2.4. *Assume $\|D^{1/2}\psi_0\|_{L^2}\le B$ where B is some fixed constant, and $\mu([0,T])\le\mu_1(B)$ where $\mu_1(B)$ is a constant that depends only on B, then we have the estimate*

$$\big\|I\psi\big\|_{X^{[0,T]}_{1,(1+\delta)/2}}\le C\big\|DI\psi_0\big\|_{L^2(R^2)}. \tag{2.23}$$

Proof. Let us consider the evolution equation below

$$i\partial_t\big(DI\psi_1\big) - \Delta\big(DI\psi_1\big) = -DI\big(\chi_{[0,T]}(t)|\psi|^2\psi\big), \tag{2.24}$$

so that $DI\psi(t) = DI\psi_1(t)$ in the interval $[0,T]$. Again we will denote $D := \mathbf{R}^2 \times [0,T]$. For Eq. (2.24) the following estimate holds true for any $\delta > 0$, see [1, 2],

$$\big\|S^{(1+\delta)/2} DI\psi_1\big\|_{L^2(D)} \le C_1\big\|DI\psi_0\big\|_{L^2} + C(\delta)\big\|S^{-(1-\delta)/2} DI\big(\chi_{[0,T]}|\psi|^2\psi\big)\big\|_{L^2(D)}, \tag{2.25}$$

where $C(\delta) \sim \delta^{-1}$. Employing Lemma 2.1 again, we have that $DI(\psi^3) \sim \psi^2(DI\psi)$. Interpolating between the Strichartz type estimate

$$\big\|\psi\big\|_{L^4_{t,x}} \le C(\delta)\big\|\psi\big\|_{X_{0,(1+\delta)/2}} \tag{2.26}$$

and the Plancherel identity, $\|\psi\|_{L^2} = \|\widehat{\psi}\|_{L^2}$, we obtain the estimate

$$\big\|\psi\big\|_{L^p_{t,x}} \le C(\delta)\big\|\psi\big\|_{X_{0,(1-\delta)/2}}, \quad \text{where} \quad p = \frac{4}{1+\theta}; \quad \theta = \frac{2\delta}{1+\delta}. \tag{2.27}$$

The dual to the estimate above reads

$$\big\|\widehat{S}^{-(1-\delta)/2}\widehat{\psi}\big\|_{L^2} \le C(\delta)\|\psi\|_{L^{p'}}; \qquad p' = \frac{4}{3-\theta} \tag{2.28}$$

and applying (2.28) on the last term of (2.25), we obtain

$$\big\|S^{-(1-\delta)/2}\big(DI\chi_{[0,T]}|\psi|^2\psi\big)\big\|_{L^2} \le C(\delta)\big\|\chi_{[0,T]}\psi^2(DI\psi)\big\|_{L^{p'}}. \tag{2.29}$$

Now an application of Holder's inequality with exponents,

$$\frac{1}{3-\theta} + \frac{2(1-\theta)}{3-\theta} + \frac{\theta}{3-\theta} = 1, \tag{2.30}$$

where the first exponents is applied to $DI\psi$, the second to $|\psi|^{8(1-\theta)/(3-\theta)}$ and the third to $|\psi|^{8\theta/(3-\theta)}$ gives

$$\big\|\psi^2(DI\psi)\big\|_{L^{p'}} \le C(\delta)\|\psi\|^{(1-\theta)/2}_{L^4(D)}\|\psi\|^{2\theta}_{L^8(D)}\|DI\psi\|_{L^4(D)}. \tag{2.31}$$

The Strichartz estimate, see Lemma 2.3, gives

$$\big\|D^{1/2}\psi\big\|_{L^{8/3}_x L^8_t(D)} \le C\big\|D^{1/2}\psi_0\big\|_{L^2} \tag{2.32}$$

provided that $\mu[0,T] \le \mu_0$ while the Sobolev embedding $W^{1/2,8/3} \subset L^8$ combined with (2.32) in (2.25) gives

$$\big\|I\psi\big\|_{X^{[0,T]}_{1,(1+\delta)/2}} \le C_1\big\|DI\psi_0\big\|_{L^2} + C(\delta)\|D^{1/2}\psi_0\|^{2\theta}_{L^2}\big[\mu^{(1-\theta)/2}([0,T])\big]\|I\psi\|_{X^{[0,T]}_{1,(1+\delta)/2}}. \tag{2.33}$$

Let us choose δ small but fixed, for example $\delta = 2^{-10}$. From the inequality (2.33), we conclude that

$$\|I\psi\|_{X^{[0,T]}_{1,(1+\delta)/2}} \le C\|DI\psi_0\|_{L^2} \tag{2.34a}$$

provided that

$$C(\delta)\|D^{1/2}\psi_0\|_{L^2}^{2\theta}\left[\mu^{(1-\theta)/2}([0,T])\right] < 1/2, \tag{2.34b}$$

which means that we would like to have $\mu([0,T])$ smaller than some fixed number, say $\mu_1(B)$ since we assumed that $\|D^{1/2}\psi_0\|_{L^2} < B$. This concludes the proof of Lemma 2.4. □

Finally the almost conservation law, see (2.13) combined with Lemma 2.4 and the scaling gives an estimate for the energy of $I\psi^{(\lambda)}$. First recall that with the scaling in (2.11), we have

$$\left\|D^{1/2}\psi_0^{(\lambda)}\right\|_{L^2} = (1/\sqrt{\lambda})\left\|D^{1/2}\psi_0\right\|_{L^2} \sim (1/CN^{1/2}) < 1 \tag{2.35a}$$

and λ was chosen so that $E(I\psi_0^{(\lambda)}) < 1/4$, so we have

$$\left|E(I\psi^{(\lambda)})(T) - E(I\psi^{(\lambda)})(0)\right| \leq \frac{C}{N^{(3-\delta)/2}}\|DI\psi_0^{(\lambda)}\|_{L^2}^4 \leq \frac{C}{N^{(3-\delta)/2}}, \tag{2.35b}$$

provided that $\mu[0,T] < \mu_1(B)$. We are now in a position to demonstrate the global in time existence of (2.1), again we will follow the argument in [6].

Proof of global existence. Start with data $\psi_0 \in C_c^\infty$ so that $\psi(t,x)$ is a global solution of our problem. Now scale the solution $\psi^{(\lambda)}$ so that $E(I\psi_0^{(\lambda)}) \leq 1/4$, i.e. the scaling parameter λ is chosen according to (2.11). The multiplier I depends on N, see (2.2), and we will choose N at the very end of the argument to be sufficiently large. For the time being let us pick T_0 arbitrarily large denote $D[0,t] := \mathbf{R}^2 \times [0,t]$ and define the following set

$$S \stackrel{\text{def}}{=} \left\{t : 0 < t \leq T_0 \text{ and } \|\psi^{(\lambda)}\|_{L^4(D[0,t])} \leq At^{1/8}\right\} \tag{2.36}$$

with A a constant to be chosen as follows

$$A \stackrel{\text{def}}{=} K\left(\|\psi_0\|_{L^2}^{3/4} + 1\right), \tag{2.37}$$

where K is some large number to be chosen later.

Claim 2.5. *We claim that $S = [0,T_0]$, i.e. the L^4 space-time norm on the slice $R^2 \times [0,t]$ is bounded by some constant times $t^{1/8}$.*

Proof of Claim 2.5: Assume not, since $\|\psi^{(\lambda)}\|_{L^4(D[0,t])}$ is a continuous function, there exist some $T \in [0,T_0]$ with the properties, $D[0,T] := \mathbf{R}^2 \times [0,T]$,

$$\|\psi^{(\lambda)}\|_{L^4(D[0,T])} > AT^{1/8} \quad \text{and} \quad \|\psi^{(\lambda)}\|_{L^4(D[0,T])} \leq 2AT^{1/8}. \tag{2.38}$$

From the correlation estimate (2.14), we know that

$$\|\psi^{(\lambda)}\|_{L^4(D[0,T)])} \leq \|\psi_0\|^{1/2}_{L^2(R^2)} \left(\sup_{[0,T]} \|\psi^{(\lambda)}(t)\|^{1/2}_{H^{1/2}(R^2)} \right) T^{1/8}, \tag{2.39}$$

and our next goal is to estimate, using the almost conservation law, the quantity below

$$\sup_{[0,T]} \|\psi^{(\lambda)}(t)\|^{1/2}_{H^{1/2}(R^2)}. \tag{2.40}$$

Let us decompose $\psi^{(\lambda)}$ in high and low frequencies, i.e. we write

$$\psi^{(\lambda)} = P_{\leq N}\left[\psi^{(\lambda)}\right] + P_{\geq N}\left[\psi^{(\lambda)}\right]. \tag{2.41}$$

This is a decomposition of $\psi^{(\lambda)}$ on frequencies $|\xi| \leq N$ and $|\xi| \geq N$ respectively. Recall that $s = 1/2$ in (2.2) and (2.3). For the low frequencies, we have, interpolating between L^2 and H^1

$$\left\|P_{\leq N}[\psi^{(\lambda)}(t)\right\|_{H^{1/2}} \leq \|P_{\leq N}[\psi^{(\lambda)}(t)\|^{1/2}_{L^2}\|P_{\leq N}[\psi^{(\lambda)}(t)\|^{1/2}_{H^1} \tag{2.42a}$$

$$\leq \|\psi_0\|^{1/2}_{L^2}\left\|\nabla I\psi^{(\lambda)}(t)\right\|^{1/2}_{L^2}. \tag{2.42b}$$

For the high frequencies, again with $s = 1/2$, we can write

$$\left\|P_{\geq N}[\psi^{(\lambda)}(t)\right\|_{H^{1/2}} \leq \frac{1}{N^{1/2}}\left\|\nabla I\psi^{(\lambda)}(t)\right\|_{L^2}. \tag{2.42c}$$

Combining the above inequalities (2.42a) and (2.42b) in (2.39), we have the overall estimate

$$\begin{aligned} \|\psi^{(\lambda)}\|_{L^4(D[0,T])} &\leq T^{1/8}\|\psi_0\|^{3/4}_{L^2} \sup_{[0,T]} \left\|\nabla I\psi^{(\lambda)}(t)\right\|^{1/4}_{L^2} \\ &\quad + T^{1/8}\frac{\|\psi_0\|^{1/2}_{L^2}}{N^{1/2}} \sup_{[0,T]} \left\|\nabla I\psi^{(\lambda)}(t)\right\|^{1/2}_{L^2}. \end{aligned} \tag{2.43}$$

Since we know from (2.38) that, $\|\psi^{(\lambda)}\|^4_{L^4(D[0,T])} \leq (2A)^4 T^{1/2}$, we can split any interval $[0,T]$, where $T \leq T_0$, in subintervals, say J_k, with $k = 1, 2, \ldots, L$, write $D_k := R^2 \times J_k$ and for each slice D_k, we have

$$\|\psi^{(\lambda)}\|^4_{L^4(D_k)} \leq \mu_1(B), \tag{2.44}$$

where $\mu_1 := min\{\mu_0, \mu_1\}$, see Lemmata 2.3 and 2.4. The number of slices, which we will call L, is at most like

$$L \sim \frac{(2A)^4 T^{1/2}}{\mu_1(B)}. \tag{2.45}$$

Because of (2.44) we can apply the almost conservation law (2.35) for each of the slices D_k. We need to control $\|D^{1/2}\psi^{(\lambda)}(T_k)\|_{L^2}$ by some constant. Observe that

$$\int |\xi||\widehat{\psi}(\xi)|^2 d\xi \leq \big(\|\psi\|_{L^2} + 1\big) E^{1/2}(I\psi) \tag{2.46}$$

and since the L^2 norm is conserved, we can bound $\|D^{1/2}\psi^{(\lambda)}(T_k)\|_{L^2}$ by a constand B, see Lemma (2.4) as long as we know that $E(I\psi^{(\lambda)}(T_k)) < 1/2$. Applying the estimate in (2.13) L times on the successive slices, we obtain

$$\sup_{[0,T]} E\big(I\psi^{(\lambda)}(t)\big) \leq E\big(I\psi_0^{(\lambda)}\big) + \frac{L}{N^{3/2-\delta}}, \tag{2.47}$$

where we know from scaling, see (2.12), that $E\big(I\psi_0^{(\lambda)}\big) \leq \frac{1}{4}$. Thus in order to guarantee that the energy of scaled function $I\psi^{(\lambda)}(t)$ satisfies $E(I\psi^{(\lambda)}(t)) < 1/2$ for all $t \in [0, T_0]$, we would like to choose

$$\frac{L}{N^{3/2-\delta}} \leq \frac{1}{4} \quad \text{or } N^{3/2-\delta} \sim 4\frac{(2A)^4 T_0^{1/2}}{\mu_1(B)}, \tag{2.48}$$

so that we have the bound

$$\sup_{[0,T]} E\big(I\psi^{(\lambda)}(t)\big) \leq 1/2 \tag{2.49}$$

and substituting (2.49) back to (2.43), we obtain

$$\|\psi^{(\lambda)}\|_{L^4(D[0,T])} \leq 2\|\psi_0\|_{L^2}^{3/4} T^{1/8}, \tag{2.50}$$

which contradicts (2.38) if $K \gg 2$. This concludes the proof of the claim and notice that the choice of N is given by (2.48). □

Finally recall that we assumed that $\psi_0 \in C_c^\infty$ hence if $\psi_0 \in H^{1/2}$ we can approximate it by $\widetilde{\psi}_0 \in C_c^\infty$ so that $\|\psi_0 - \widetilde{\psi}_0\|_{H^{1/2}} \to 0$. Since we considered the solution in an arbitrarily large interval of time $[0, T_0]$ and the proof of Claim 2.5 involves only the $H^{1/2}$ norm of the solution we obtain existence of solutions for arbitrarily large time. This concludes the proof of global existence. □

3. Bourgain's Quadratic Estimate

Let us start by explaining a quadratic estimate derived by Bourgain [1, 2], which estimate is crucial in deriving the almost conservation law in the next section. The main estimate is stated in (3.11) and rephrased in (3.14) in a manner which is convenient for our purpose. Consider two functions $\psi_1(t, \mathbf{x})$ and $\psi_2(t, \mathbf{x})$ and their corresponding Fourier transforms, say $\widehat{\psi}_1(\tau, \xi)$ and $\widehat{\psi}_2(\tau, \xi)$ respectively. Let us write the Fourier transform of the product $\psi_1\psi_2$

$$\widehat{\psi_1\psi_2}(\tau, \xi) = \int_{R\times R^2} \widehat{\psi}_1(\tau - \sigma, \xi - \eta)\widehat{\psi}_2(\sigma, \eta) d\eta d\sigma. \tag{3.1}$$

The idea is to rewrite (3.1) in parabolic variables, let us define first the new variables,

$$u \stackrel{\text{def}}{=} \tau - \sigma - |\xi - \eta|^2; \quad v \stackrel{\text{def}}{=} \sigma - |\eta|^2 \tag{3.2}$$

and write $\psi_2^\sharp(v, \eta)$ and $\psi_1^\sharp(u, \xi - \eta)$ for the functions $\widehat{\psi}_2$ and $\widehat{\psi}_1$ respectively. At this point it is convenient to define

$$p \stackrel{\text{def}}{=} |\xi - \eta|^2 + |\eta|^2; \quad \eta_1 \stackrel{\text{def}}{=} \eta - \xi/2, \tag{3.3}$$

so that we can write

$$|\eta_1|^2 = \left(2p - |\xi|^2\right)/4. \tag{3.4}$$

Let us call $\rho := |\eta_1|$ the length of the vector η_1 so that we can write $\eta_1 = \rho e^{i\alpha}$, where α is the angle of the vector η_1 and express ρ as a function of p and ξ via the formula

$$\rho(p, \xi) = \frac{1}{2}\sqrt{2p - |\xi|^2}. \tag{3.5}$$

Notice that $d\eta = d\eta_1 = \rho d\rho d\alpha = (1/4) dp d\alpha$, moreover we have that $p = \tau - u - v$, so that we can write ρ as a function of (τ, u, v, ξ)

$$\rho(\tau, u, v, \xi) = \frac{1}{2}\sqrt{2(\tau - u - v) - |\xi|^2}. \tag{3.6}$$

Combining all these we can write

$$\left(\psi_1\psi_2\right)(t, \mathbf{x}) = \mathcal{F}^{-1}\left\{\Phi(\tau, \xi)\right\}, \tag{3.7a}$$

where $\Phi(\tau, \xi)$ is given by the following integral

$$\Phi(\tau, \xi) \stackrel{\text{def}}{=} \int_{S^1 \times R \times R} \psi_1^\sharp(u, \xi/2 - \rho e^{i\alpha}) \psi_2^\sharp(v, \xi/2 + \rho e^{i\alpha})\, d\alpha du dv \tag{3.7b}$$

and $\rho(u, v, \xi)$ is given by the formula (3.6). Now let us consider a dyadic decomposition of ψ_1 and ψ_2 as follows

$$u \sim 2^{l_1}; \quad v \sim 2^{l_2}; \quad |\xi| \sim 2^{j_1}; \quad |\eta| \sim 2^{j_2} \tag{3.8}$$

and assume without loss of generality that $2^{j_1} \gg 2^{j_2}$. The crucial observation is that since

$$\xi/2 - \rho e^{i\alpha} \sim 2^{j_1}; \quad \xi/2 + \rho e^{i\alpha} \sim 2^{j_2}; \quad 2^{j_1} \gg 2^{j_2}, \tag{3.9}$$

we have that the angle α of the vector η_1 is constrained in an interval $[-\alpha_0, \alpha_0]$, where the size of the interval is restricted by

$$\alpha_0 \sim \frac{2^{j_2}}{2^{j_1}}. \tag{3.10}$$

Plancherel in (3.7a) and (3.7b) gives the estimate

$$\left\|\psi_1\psi_2\right\|_{L^2}^2 \le C 2^{j_2 - j_1} 2^{l_1} 2^{l_2} \|\psi_1\|_{L^2}^2 \|\psi_2\|_{L^2}^2. \tag{3.11}$$

The estimate above can be expressed in a form which is more convenient for our purpose. Let us define first,

$$\langle s\rangle \stackrel{\text{def}}{=} |s|+1; \quad \widehat{S} \stackrel{\text{def}}{=} \langle \tau - |\xi|^2\rangle \tag{3.12a}$$

$$b(\xi_1,\xi_2) \stackrel{\text{def}}{=} \min\left\{\left(\frac{\langle\xi_1\rangle}{\langle\xi_2\rangle}\right)^{1/2}, \left(\frac{\langle\xi_2\rangle}{\langle\xi_1\rangle}\right)^{1/2}\right\}. \tag{3.12b}$$

Using the function b we can define the quadratic expression, for $\delta > 0$, arbitrarily small

$$b^{(-1+\delta)}(\psi_1\psi_2) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\left\{\int b^{(-1+\delta)}(\xi-\eta,\eta)\widehat{\psi}_1(\tau-\sigma,\xi-\eta)\widehat{\psi}_2(\sigma,\eta)d\sigma d\eta\right\}, \tag{3.13}$$

and the estimate in (3.11) implies the following estimate for the quadratic expression in (3.13)

$$\left\|b^{(-1+\delta)}(\psi_1\psi_2)\right\|_{L^2} \le C(\delta)\left\|\widehat{S}^{(1+\delta)/2}\widehat{\psi}_1\right\|_{L^2}\left\|\widehat{S}^{(1+\delta)/2}\widehat{\psi}_2\right\|_{L^2}; \quad \delta > 0. \tag{3.14}$$

Notice that from the definition in (3.12b), we have that $b^{-1+\delta} \ge 1$.

4. Almost Conservation Law

The almost conservation law was developed in a series of papers by the authors of [5, 6], for the Schrodinger equation, see [5, 6]. Presently we would like to rederive the almost conservation law. The derivation is slightly different but we follow the same reasoning. Start by writing the evolution equation for $\psi_I := I\psi$, where $I\psi := \mathcal{F}^{-1}(m\psi)$,

$$i\partial_t\psi_I - \Delta\psi_I + |\psi_I|^2\psi_I + \left(I\left(|\psi|^2\psi\right) - |\psi_I|^2\psi_I\right) = 0. \tag{4.1}$$

The energy of ψ_I is defined to be, see (2.7),

$$E\left(\psi_I\right) \stackrel{\text{def}}{=} \int_{R^2}\left\{\left|\nabla\left(\psi_I\right)\right|^2 + \frac{1}{2}\left|\psi_I\right|^4\right\}d\mathbf{x}, \tag{4.2}$$

in imitation of the conserved energy. The energy in (4.2) is not conserved but in the terminology of [5, 6] it is almost conserved. Let us define two quantities

$$P \stackrel{\text{def}}{=} I\left(|\psi|^2\psi\right); \quad R \stackrel{\text{def}}{=} |\psi_I|^2\psi_I, \tag{4.3}$$

so that from the evolution Eq. (4.1), we can derive an evolution equation for $E(I\psi)$, namely

$$\frac{dE}{dt} = \left\langle P-R\ ;\ \left(\overline{\psi}_I\right)_t\right\rangle_{L^2} + \left\langle\overline{P}-\overline{R}\ ;\ \left(\psi_I\right)_t\right\rangle_{L^2}. \tag{4.4}$$

Using the evolution equation, we can write

$$\phi \stackrel{\text{def}}{=} \partial_t\psi = -i\Delta\psi + i\left(|\psi|^2\psi\right), \tag{4.5a}$$

so that we can split $\phi = \phi_1 + \phi_2$ where we define

$$\phi_1 \stackrel{\text{def}}{=} -i\Delta\psi; \quad \phi_2 \stackrel{\text{def}}{=} i\left(|\psi|^2\psi\right). \tag{4.5b}$$

Notice that $I\phi_2 = iP$ so if we substitute back in (4.4), we obtain

$$\frac{dE}{dt} = 2\mathrm{Im}\left\langle P - R\ ;\ I\overline{\phi}_1\right\rangle_{L^2} + 2\mathrm{Im}\left\langle P - R\ ;\ \overline{R}\right\rangle_{L^2}. \tag{4.6}$$

The almost conservation of energy means that we would like to prove the estimate

$$\left|E(T) - E(0)\right| \le N^{-(3-\delta)/2}\left(C_1\, \|I\psi\|^4_{X^{[0,T]}_{1,\alpha}} + C_2\, \|I\psi\|^6_{X^{[0,T]}_{1,\alpha}}\right); \quad \alpha = (1+\delta)/2. \tag{4.7}$$

Lemma 4.1. *Assume* $1/2 \le s < 1$. *For the quantity defined below*

$$J(t) \stackrel{\text{def}}{=} \left\langle I_s\left(|\psi|^2\psi\right) - |I_s\psi|^2 I_s\psi\ ;\ \phi\right\rangle, \tag{4.8a}$$

the following estimate holds true

$$\left|\int_0^T J(t)dt\right| \le \frac{C}{N^{(3-\delta)/2}}\|I_{s_1}\psi\|^3_{X^{[0,T]}_{1,\alpha}}\|I_\beta\phi\|_{X^{[0,T]}_{-1,\alpha}}; \quad \alpha = \frac{1+\delta}{2}, \tag{4.8b}$$

where s_1 *and* β *are given by*

$$s_1 = s - \frac{1-\delta}{4}; \quad \beta = \frac{3-\delta}{4}; \quad s \ge \frac{1}{2}; \quad \delta > 0. \tag{4.8c}$$

At this point let us notice that applying the lemma above to $I\phi_1$, see (4.5) and the first term on the right-hand side of (4.6), we obtain since $s_1 < s_2$ imply $m_{s_1} \le m_{s_2}$,

$$\left|\int_0^T \langle P - R\ ;\ I\phi_1\rangle dt\right| \le \frac{C}{N^{(3-\delta)/2}}\|I\psi\|^3_{X^{[0,T]}_{1,\alpha}}\|I\phi_1\|_{X^{[0,T]}_{-1,\alpha}} \le \frac{C}{N^{(3-\delta)/2}}\|I\psi\|^4_{X^{[0,T]}_{1,\alpha}}. \tag{4.9}$$

Proof. We will ignore complex conjugates since they are irrelevant here and also ignore the dependence of functions on the time variable. Writing the integral above in Fourier space, we obtain

$$J = \int \left[1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right]\widehat{\psi}_I(\xi_2)\widehat{\psi}_I(\xi_3)\widehat{\psi}_I(\xi_4)\widehat{\phi}(\xi_2+\xi_3+\xi_4)\ d\xi_2 d\xi_3 d\xi_4. \tag{4.10a}$$

Let us write for simplicity,

$$m_{s,N}(\xi) = \begin{cases} 1, & \text{if } |\xi| < N, \\ (|\xi|/N)^{1-s}, & \text{if } |\xi| \ge N, \end{cases} \tag{4.10b}$$

and call $\xi_1 := \xi_2 + \xi_3 + \xi_4$. Rewrite the integral appearing in (4.10a) in the following manner

$$J = \int \left[1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right]\frac{|\xi_1|}{|\xi_2||\xi_3||\xi_4|} \times |\xi_2|\widehat{\psi}_I(\xi_2)|\xi_3|\widehat{\psi}_I(\xi_3)|\xi_4|\widehat{\psi}_I(\xi_4)|\xi_1|^{-1}\widehat{\phi}(\xi_1)d\xi_2 d\xi_3 d\xi_4. \tag{4.10c}$$

Employing the notation $b_{2,4} := b(\xi_2, \xi_4)$ and $b_{1,3} := b(\xi_1, \xi_3)$ for the expression b defined in (3.12a and 3.12b) and keeping in mind the estimate in (3.14), write the integral J in the following way

$$J = \int \left[\frac{|\xi_1|}{|\xi_2||\xi_3||\xi_4|} - \frac{a(\xi_1)}{a(\xi_2)a(\xi_3)a(\xi_4)} \right] b_{2,4}^{1-\delta}(\xi_2,\xi_4) b_{1,3}^{1-\delta}(\xi_1,\xi_3) \times \left(|\xi_2|\widehat{\psi}_I(\xi_2)|\xi_4|\widehat{\psi}_I(\xi_4) b_{2,4}^{-1+\delta} \right) \left(|\xi_1|^{-1}\widehat{\phi}(\xi_1)|\xi_3|\widehat{\psi}_I(\xi_3) b_{1,3}^{-1+\delta} \right) d\xi_2 d\xi_3 d\xi_4, \tag{4.10d}$$

where $a(\xi)$ stands for the function

$$a(\xi) = |\xi| m_{s,N}(\xi) = \begin{cases} |\xi|, & \text{if } |\xi| \le N, \\ N^{1-s}|\xi|^s, & \text{if } |\xi| > N. \end{cases} \tag{4.10e}$$

Give a name to the first part of the integrand in (4.10d),

$$Q \stackrel{\text{def}}{=} \left[\frac{|\xi_1|}{|\xi_2||\xi_3||\xi_4|} - \frac{a(\xi_1)}{a(\xi_2)a(\xi_3)a(\xi_4)} \right] b_{2,4}^{1-\delta}(\xi_2,\xi_4) b_{1,3}^{1-\delta}(\xi_1,\xi_3). \tag{4.11}$$

The desired estimate in (4.8b) will follow from the quadratic estimates in (3.14) if we demonstrate the bound below.

Claim 4.2. *The quantity Q defined in* (4.11) *satisfies the bound*

$$|Q| \le N^{-(3-\delta)/2} (\lambda_1\lambda_2\lambda_3\lambda_4)^{(1-\delta)/4}; \quad \delta > 0, \tag{4.12a}$$

where λ_j are the quantities

$$\lambda(\xi) = \begin{cases} 1, & \text{if } |\xi| \le N, \\ N/|\xi|, & \text{if } |\xi| \ge N; \end{cases} \quad \lambda_j = \lambda(\xi_j). \tag{4.12b}$$

Proof of Claim 4.2. We can assume without loss of generality that $|\xi_4| \le |\xi_3| \le |\xi_4|$ and write

$$\rho_j = |\xi_j|; \quad j = 1,2,3,4, \tag{4.13}$$

so that $\rho_4 \le \rho_3 \le \rho_2$. First let us observe that if $\rho_4 \le \rho_3 \le \rho_2 \le N$ and $\rho_1 \le N$ then $Q = 0$. We will examine the size of the quantity Q in three different cases.

Case 1. Assume that $\rho_4 \le \rho_3 \le N \le \rho_2$. The quantity Q is largest when $\rho_4 \le \rho_3 \ll N$ hence $\rho_1 \approx \rho_2$. In this case we have,

$$\begin{aligned} Q &= \left(\frac{\rho_1}{\rho_2\rho_3\rho_4} - \frac{\rho_1^s}{\rho_2^s\rho_3\rho_4} \right) \left(\frac{\rho_4}{\rho_2} \right)^{(1-\delta)/2} \left(\frac{\rho_3}{\rho_1} \right)^{(1-\delta)/2} \\ &= \left[\left(1 + \frac{\rho_1-\rho_2}{\rho_2} \right) - \left(1 + \frac{\rho_1-\rho_2}{\rho_2} \right)^s \right] \frac{1}{(\rho_1\rho_2)^{(1-\delta)/2}(\rho_3\rho_4)^{(1+\delta)/2}} \\ &\approx \frac{\rho_1-\rho_2}{\rho_2(\rho_1\rho_2)^{(1-\delta)/2}(\rho_3\rho_4)^{\frac{1+\delta}{2}}} \le \frac{\rho_3+\rho_4}{(\rho_3\rho_4)^{(1+\delta)/2}} \left(\frac{N}{\rho_2} \right)^{\frac{3-\delta}{2}} \left(\frac{N}{\rho_1} \right)^{\frac{1-\delta}{2}} N^{-2+\delta}, \end{aligned} \tag{4.14a}$$

where we used the fact $|\rho_1-\rho_2| \le \rho_3+\rho_4 \ll N$ and $(1+x)^s-1 \le Cx$ for $x \in (0,1)$. The estimate in (4.14a) can be expressed in the following manner

$$\big|Q\big| \le C\lambda_2^{(3-\delta)/2}\lambda_1^{(1-\delta)/2}N^{-2+\delta}. \tag{4.14b}$$

Case 2. Assume $\rho_4 \le N \le \rho_3 \le \rho_2$. The quantity Q is largest when $\rho_3 \approx N$, $\rho_4 \ll N$, hence $\rho_1 \approx \rho_2$. In this case we have the estimates

$$\begin{aligned} Q &= \left(\frac{\rho_1}{\rho_2\rho_3\rho_4} - \frac{N^{1-s}\rho_1^s}{N^{2(1-s)}(\rho_2\rho_3)^s\rho_4}\right)\left(\frac{\rho_4\rho_3}{\rho_2\rho_1}\right)^{(1-\delta)/2} \\ &\approx \left(\frac{\rho_1}{\rho_2\rho_3}\right)^s\frac{1}{\rho_4}\left|\left(\frac{\rho_1}{\rho_2\rho_3}\right)^{1-s} - \frac{1}{N^{1-s}}\right|\left(\frac{\rho_3\rho_4}{\rho_1\rho_2}\right)^{(1-\delta)/2} \\ &\le \left(\frac{\rho_1}{\rho_2\rho_3}\right)^s\left(\frac{\rho_3\rho_4}{\rho_1\rho_2}\right)^{(1-\delta)/2}\frac{1}{N^{1-s}} \approx \left(\frac{N}{\rho_3}\right)^{s-(1-\delta)/2}\frac{N^2}{(\rho_1\rho_2)^{\frac{1-\delta}{2}}}N^{\frac{-3+\delta}{2}}. \end{aligned} \tag{4.15a}$$

Using the notation from (4.12b) we can write (4.15a) as follows

$$\big|Q\big| \le C\big(\lambda_1\lambda_2\big)^{(1-\delta)/2}\lambda_3^{s-(1-\delta)/2}N^{-(3-\delta)/2}. \tag{4.15b}$$

Case 3. Assume $N \le \rho_4 \le \rho_3 \le \rho_2$. The quantity Q is largest when $\rho_4 \le \rho_3 \ll \rho_2$, hence $\rho_1 \approx \rho_2$. Now we can estimate

$$\begin{aligned} Q &= \left[\frac{\rho_1}{\rho_2\rho_3\rho_4} - \frac{\rho_1^s}{N^{2(1-s)}(\rho_2\rho_3\rho_4)^s}\right]\left(\frac{\rho_3\rho_4}{\rho_1\rho_2}\right)^{(1-\delta)/2} \\ &= \left(\frac{\rho_1}{\rho_2\rho_3\rho_4}\right)^s\left[\left(\frac{\rho_1}{\rho_2\rho_3\rho_4}\right)^{1-s} - \frac{1}{N^{2(1-s)}}\right]\left(\frac{\rho_3\rho_4}{\rho_1\rho_2}\right)^{(1-\delta)/2} \\ &\le \left(\frac{\rho_1}{\rho_2\rho_3\rho_4}\right)^s\frac{1}{N^{2(1-s)}}\left(\frac{\rho_3\rho_4}{\rho_1\rho_2}\right)^{(1-\delta)/2} \\ &\le \left(\frac{\rho_1}{\rho_2}\right)^s\left(\frac{N}{\rho_3}\right)^{s-(1-\delta)/2}\left(\frac{N}{\rho_2}\right)^{1/2}\left(\frac{N}{\rho_1}\right)^{1/2}\left(\frac{N}{\rho_4}\right)^{s-(1-\delta)/2}N^{-2}, \end{aligned} \tag{4.16a}$$

which can be written, using the notation in (4.12b)

$$\big|Q\big| \le \big(\lambda_3\lambda_4\big)^{s-(1-\delta)/2}\big(\lambda_1\lambda_2\big)^{(1-\delta)/2}N^{-2}. \tag{4.16b}$$

The three cases demonstrate the inequality in (4.12a) i.e. we see that for $s \ge 1/2$, we have that the quantity Q is bounded by

$$\big|Q\big| \le CN^{-(3-\delta)/2}\left(\lambda_1\lambda_2\lambda_3\lambda_4\right)^{(1-\delta)/4}, \tag{4.17}$$

where $\lambda_j := \lambda(\xi_j)$ are given by (4.12b). This concludes the proof of the claim.

Hence this completes the proof of Lemma 4.1. □

Let us now apply the estimate in Lemma 4.1 for the expression below, see the second term on the right-hand side of (4.6)

$$\left|\int_0^T \langle P-R;\ R\rangle\, dt\right| \le \frac{C}{N^{(3-\delta)/2}} \|I\psi\|^3_{X^{[0,T]}_{1,\alpha}} \|I_\beta R\|_{X^{[0,T]}_{-1,\alpha}}, \tag{4.18a}$$

so that we would like to estimate the expression

$$\left\|\left(\widehat{S}^\alpha/\langle\xi\rangle\right) I_\beta \widehat{R}\right\|_{L^2}; \quad R = |\psi_I|^2\psi_I. \tag{4.18b}$$

For simplicity we can write

$$\widehat{S}_1 \stackrel{\text{def}}{=} \left\langle \tau_2+\tau_3+\tau_4-\left|\xi_2+\xi_3+\xi_4\right|^2\right\rangle; \quad \widehat{S}_k \stackrel{\text{def}}{=} \left\langle \tau_k - \left|\xi_k\right|^2\right\rangle, \quad k=2,3,4 \tag{4.19a}$$

and, denoting $\alpha := (1+\delta)/2$, we have the following inequality

$$\widehat{S}_1^\alpha \le \widehat{S}_2^\alpha + \widehat{S}_3^\alpha + \widehat{S}_4^\alpha + \left(|\xi_2||\xi_3|\right)^\alpha + \left(|\xi_3||\xi_4|\right)^\alpha + \left(|\xi_2||\xi_4|\right)^\alpha. \tag{4.19b}$$

With these observations in mind we have that as far as the L^2 norm is concerned, we can write

$$\frac{m_\beta(\xi)}{\langle\xi\rangle}\left(\widehat{S}^\alpha \widehat{|\psi_I|^2\psi_I}\right) \approx \frac{m_\beta(\xi)}{\langle\xi\rangle}\widehat{\psi_I^2} * \left(\widehat{S}^\alpha\widehat{\psi_I}\right) + \frac{m_\beta(\xi)}{\langle\xi\rangle}\widehat{\psi_I} * \left(|\xi|^\alpha\widehat{\psi_I}\right) * \left(|\xi|^\alpha\widehat{\psi_I}\right). \tag{4.20}$$

Combining the Strichartz type estimate in (2.18a) and (2.18b) with the Sobolev embedding, we obtain the following inequalities

$$\|D\psi_I\|_{L_x^2L_t^\infty} + \left\|\psi_I\right\|_{L_x^\infty L_t^r} \le C\left\|\psi_I\right\|_{X_{1,\alpha}}; \quad 2<r, \tag{4.21a}$$

$$\left\|S^\alpha\psi_I\right\|_{L_x^qL_t^2} \le C\left\|\psi_I\right\|_{X_{1,\alpha}}; \quad q\ge 2, \tag{4.21b}$$

$$\left\|D^\alpha\psi_I\right\|_{L_x^pL_t^4} \le C\left\|\psi_I\right\|_{X_{1,\alpha}}; \quad p\le 4/\delta. \tag{4.21c}$$

Finally employing the inequalities in (4.21a)–(4.21c), we have for each of the terms appearing on the right-hand side of (4.20) and for any $p>1$ and $(1/r)+(1/r')=1$

$$\left\|\langle\xi\rangle^{-1}\mathcal{F}\left(\psi_I^2S^\alpha\psi_I\right)\right\|_{L^2} \le C\left\|\psi_I\right\|^2_{L_x^{2pr'}L_t^\infty}\left\|S^\alpha\psi_I\right\|_{L_x^{pr}L_t^2} \le C\left\|\psi_I\right\|^3_{X_{1,\alpha}}, \tag{4.22a}$$

$$\left\|\langle\xi\rangle^{-1}\mathcal{F}\left(\psi_I\left(D^\alpha\psi_I\right)^2\right)\right\|_{L^2} \le C\left\|\psi_I\right\|_{L_x^{pr'}L_t^\infty}\left\|D^\alpha\psi_I\right\|^2_{L_x^{2pr}L_t^4} \le C\left\|\psi_I\right\|^3_{X_{1,\alpha}}. \tag{4.22b}$$

Combining (4.22a) and (4.22b) we obtain

$$\left|\int_0^T \langle P-R\ ;\ R\rangle dt\right| \le \frac{C}{N^{(3-\delta)/2}}\|I\psi\|^6_{X^{[0,T]}_{1,\alpha}}, \tag{4.23}$$

and this estimate together with (4.9) imply the almost concervation law in (4.7).

5. Correlation Estimates

We would like to describe a general method for deriving correlation estimates for Schrödinger type equations. The estimate which is relevant here is stated in (5.40). The idea is to view the evolution equation as describing the evolution of a compressible dispersive fluid whose pressure is a function of the density. For simplicity we will consider a cubic equation, but the method works for more general nonlinearities. Let us start by considering the Nonlinear Schrödinger equation with a defocusing cubic nonlinearity, i.e. the field $\psi(t, \mathbf{x})$ satisfies the equation

$$i\psi_t - \Delta\psi + |\psi|^2\psi = 0; \quad \psi : R \times R^n \mapsto C, \tag{5.1a}$$

where the space dimension is $n = 2, 3$ and C is the complex plane. The cubic nonlinearity serves as an example, one can consider more general nonlinear terms as long as they are of defocusing type. Let us adopt the conventions

$$D[T_1, T_2] \stackrel{\text{def}}{=} R^n \times [T_1, T_2]; \quad R^n_t \stackrel{\text{def}}{=} \left\{(t, \mathbf{x}) \ : \ \mathbf{x} \in R^n\right\} \tag{5.1b}$$

to denote a space-time slab and a time slice for t fixed. A central observation is that the Eq. (5.1a) above conserves energy which means that the integral

$$E \stackrel{\text{def}}{=} \int_{R^n_t} \left\{|\nabla\psi|^2 + \frac{1}{2}|\psi|^2\right\} \, d\mathbf{x} \tag{5.1c}$$

is a constant independent of time. Equation (5.1a) has more conservation laws certain of which we wish to exploit in order to obtain apriori estimates. Let us define certain quantities that will be central in our investigation,

$$\rho \stackrel{\text{def}}{=} \frac{1}{2}|\psi|^2, \tag{5.2a}$$

$$p_j \stackrel{\text{def}}{=} \frac{1}{2i}\left(\overline{\psi}\nabla_j\psi - \psi\nabla_j\overline{\psi}\right); \quad j = 1, 2, \ldots, n, \tag{5.2b}$$

$$p_0 \stackrel{\text{def}}{=} \frac{1}{2i}\left(\overline{\psi}\psi_t - \psi\overline{\psi}_t\right), \tag{5.2c}$$

$$\sigma_{jk} \stackrel{\text{def}}{=} \frac{1}{2}\left(\nabla_j\psi\nabla_k\overline{\psi} + \nabla_j\overline{\psi}\nabla_k\psi\right); \quad j, k = 1, 2, \ldots, n. \tag{5.2d}$$

What I defined above is a density function $\rho(t, \mathbf{x})$, a momentum vector $p_j(t, \mathbf{x})$, with p_0 the time component, and a stress tensor $\sigma_{jk}(t, \mathbf{x})$. The Schrödinger evolution gives conservation laws for the density and the space components of the momenta, they are

$$\partial_t\{\rho\} - \nabla_j\{p^j\} = 0, \tag{5.3a}$$

$$\partial_t\{p_j\} - \nabla_k\{\mu^k_{\ j}\} = 0, \tag{5.3b}$$

where $\mu_{kj}(t, \mathbf{x})$ is the tensor

$$\mu_{jk} \stackrel{\text{def}}{=} 2\sigma_{kj} - \delta_{kj}\left(\Delta\rho - 2\rho^2\right). \tag{5.3c}$$

The tensor μ_{kj} consists of three parts, the term $-\Delta\rho$ is dispersive while the term $2\rho^2$ describes the pressure as a function of density. Some remarks are in order, Eq. (5.1a) is the evolution equation of the Lagrangian

$$\int_{R^n \times R} \left[-p_0 + \sigma + 2\rho^2 \right] \, d\mathbf{x} dt, \tag{5.4a}$$

where $\sigma = tr(\sigma_{jk})$. The integral of the quantity p_0 is not preserved by the flow, but the integral

$$\int_{R^n_t} \left[-p_0 + 2\rho^2 \right] \, d\mathbf{x}$$

is, indeed, constant independent of time. This observation follows from a structure equation that (5.1a) satisfies, namely

$$-p_0 - \Delta\rho + \sigma + 4\rho^2 = 0. \tag{5.4b}$$

Notice that the system (5.3a) and (5.3b) describes the conservation laws of an irrotational compressible and dispersive fluid.

Let us assume that we have two different solutions of (5.1a) for which we have the corresponding densities ρ^a and momenta $\mathbf{p}^a$ where $a = 1, 2$. Now write the conservation law for the momenta again emphasizing the space and time dependence

$$\partial_t \left\{ p^1_j(t,\mathbf{x}) \right\} - \nabla_k \left\{ \mu^{1,k}_{\;j}(t,\mathbf{x}) \right\} = 0. \tag{5.5}$$

Multiplying the equation above with $\rho^2(t,\mathbf{y})$ and using the conservation of mass equation we obtain an equation, of the form

$$\partial_t \left\{ p^1_j(t,\mathbf{x})\rho^2(t,\mathbf{y}) \right\} - \nabla_{\mathbf{y},k} \left\{ p^1{}_j(t,\mathbf{x}) p^{2,k}(t,\mathbf{y}) \right\} - \nabla_{\mathbf{x},k} \left\{ \mu^{1,k}_{\;j}(t,\mathbf{x})\rho^2(t,\mathbf{y}) \right\} = 0. \tag{5.6}$$

In Eq. (5.6) above, we used the convention $\nabla_{\mathbf{x}}$, $\nabla_{\mathbf{y}}$ to keep track of which variable is differentiated. Change variables as follows

$$\mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{y}}{2}; \quad \mathbf{y} \mapsto \frac{\mathbf{x}-\mathbf{y}}{2} \tag{5.7}$$

and symmetrize in the $a = 1, 2$ indices, i.e. switch the roles of 1 and 2 and add them so that Eq. (5.6) can be written as

$$\partial_t \left\{ Q^{(12)}_j(t,\mathbf{x};\mathbf{y}) \right\} - \nabla_{\mathbf{y},k} \left\{ M^{(12),k}_{\;j}(t,\mathbf{x};\mathbf{y}) \right\} - \nabla_{\mathbf{x},k} \left\{ N^{(12),k}_{\;j}(t,\mathbf{x};\mathbf{y}) \right\} = 0, \tag{5.8}$$

where we use the convention (12) to denote the symmetrized quantities and the quantities $Q^{(12)}_j$, $M^{(12)}_{kj}$ and $N^{(12)}_{kj}$ are defined below,

$$Q^{(12)}_j(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} p^{(1}_j(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y}), \tag{5.9a}$$

$$M^{(12)}_{kj}(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} \mu^{(1}_{jk}(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y}) - p^{(1}_j(t,\mathbf{x}+\mathbf{y})p^{2)}_k(t,\mathbf{x}-\mathbf{y}), \tag{5.9b}$$

$$N^{(12)}_{kj}(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} \mu^{(1}_{jk}(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y}) + p^{(1}_j(t,\mathbf{x}+\mathbf{y})p^{2)}_k(t,\mathbf{x}-\mathbf{y}). \tag{5.9c}$$

In what follows we will occasionally skip the indices $a = 1, 2$ for the two solutions of (5.1a) since it simplifies the appearance of the formulas without creating much confusion. If we switch $\mathbf{y} \mapsto -\mathbf{y}$, we obtain another equation like (5.8) with the sign of the second term switched. Subtracting the two equations we obtain a symmetrized version of (5.8), which reads

$$\partial_t \{Q_{A,j}\} - \nabla_{\mathbf{y},k} \left\{M^k_{S,j}\right\} - \nabla_{\mathbf{x},k} \left\{N^k_{A,j}\right\} = 0, \tag{5.10}$$

where the relevant symmetrized or anti-symmetrized quantities are defined below

$$Q_{A,j}(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} Q_j(t,\mathbf{x};\mathbf{y}) - Q_j(t,\mathbf{x};-\mathbf{y}), \tag{5.11a}$$

$$M_{S,kj}(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} M_{kj}(t,\mathbf{x};\mathbf{y}) + M_{kj}(t,\mathbf{x};-\mathbf{y}), \tag{5.11b}$$

$$N_{A,kj}(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} N_{kj}(t,\mathbf{x};\mathbf{y}) - N_{kj}(t,\mathbf{x};-\mathbf{y}). \tag{5.11c}$$

Notice that a similar argument, starting from the conservation of mass Eq. (5.3a) gives the equation

$$\partial_t\{D(t,\mathbf{x};\mathbf{y}\} - \nabla_{\mathbf{y},k} \left\{Q^k_A(t,\mathbf{x};\mathbf{y})\right\} - \nabla_{\mathbf{x},k} \left\{Q^k_S(t,\mathbf{x};\mathbf{y})\right\} = 0, \tag{5.12a}$$

where $D(t,\mathbf{x};\mathbf{y})$ and $Q_{S,k}(t,\mathbf{x};\mathbf{y})$ are defined as follows

$$D^{(12)}(t,\mathbf{x};\mathbf{y}) = \rho^{(1}(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y}), \tag{5.12b}$$

$$Q^{(12)}_{S,k}(t,\mathbf{x};\mathbf{y}) = p^{(1}_k(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y}) + p^{(1}_k(t,\mathbf{x}-\mathbf{y})\rho^{2)}(t,\mathbf{x}+\mathbf{y}). \tag{5.12c}$$

Symmetrizing Eq. (5.8) gives an equation similar to (5.10)

$$\partial_t \{Q_{S,j}\} - \nabla_{\mathbf{y},k} \left\{M^k_{A,j}\right\} - \nabla_{\mathbf{x},k} \left\{N^k_{S,j}\right\} = 0, \tag{5.13}$$

where $M_{A,jk}$ and $N_{S,jk}$ stand for the anti-symmetrized and symmetrized versions of M_{jk} and N_{jk} respectively but we are not going to make use of (5.13) here.

A crucial observation is the fact that we can express the stress tensor σ_{kj} in terms of the density function ρ and the momenta p_j, namely an elementary calculation, using the fact that

$$\nabla_j \rho = \frac{1}{2}\left(\overline{\psi}\nabla_j\psi + \psi\nabla_j\overline{\psi}\right)$$

shows that we can write

$$\sigma_{kj} = \frac{1}{2\rho}\left[\nabla_k\rho\nabla_j\rho + p_k p_j\right]. \tag{5.14}$$

The equation above suggests that we split σ_{kj} in two parts, a potential part say π_{kj} and a kinetic part, say κ_{kj}, in the following manner

$$\pi_{kj} \stackrel{\text{def}}{=} \frac{1}{2\rho}\nabla_k\rho\nabla_j\rho; \qquad \kappa_{kj} \stackrel{\text{def}}{=} \frac{1}{2\rho}p_k p_j. \tag{5.15}$$

The trace of the potential part of the stress tensor $\mathrm{tr}(\pi_{kj})$ contains part of the potential energy of the field ψ due to compressibility, while the trace of the kinetic part $\mathrm{tr}(\kappa_{kj})$ contains the kinetic energy of the field. Our next step is to examine carefully the tensor $M_{S,kj}$. We would like to suppress the dependence on t since it is not relevant in what follows. If we substitute (5.15) in the expression of μ_{kj}, we can write a long formula for $M_{S,kj}$ as follows

$$\begin{aligned} M^{(12)}_{S,kj} &= -p^{(1}_j(\mathbf{x}+\mathbf{y})p^{2)}_k(\mathbf{x}-\mathbf{y}) - p^{(1}_j(\mathbf{x}-\mathbf{y})p^{2)}_k(\mathbf{x}+\mathbf{y}) \\ &\quad + 2\kappa^{(1}_{kj}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + 2\kappa^{(1}_{kj}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y}) \\ &\quad + 2\pi^{(1}_{kj}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + 2\pi^{(1}_{kj}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y}) \\ &\quad - \delta_{kj}[\Delta\rho^{(1}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + \Delta\rho^{(1}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y})] \\ &\quad + \delta_{kj}[2(\rho^{(1})^2(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + 2(\rho^{(1})^2(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y})]. \end{aligned} \tag{5.16}$$

Let us define two vectors, using the momenta and density

$$J^{1,2}_j(t,\mathbf{x};\mathbf{y}) \stackrel{\text{def}}{=} \frac{p^1_j(\mathbf{x}+\mathbf{y})}{\sqrt{\rho^1(\mathbf{x}+\mathbf{y})}}\sqrt{\rho^2(\mathbf{x}-\mathbf{y})} - \frac{p^2_j(\mathbf{x}-\mathbf{y})}{\sqrt{\rho^2(\mathbf{x}-\mathbf{y})}}\sqrt{\rho^1(\mathbf{x}+\mathbf{y})} \tag{5.17}$$

and similarly $J^{2,1}_j$ by switching the roles of $1, 2$ so that the first four terms in (5.14) can be expressed as a tensor product of the vector $J^{a,b}_j$, i.e. we can write

$$\begin{aligned} &-p^{(1}_j(\mathbf{x}+\mathbf{y})p^{2)}_k(\mathbf{x}-\mathbf{y}) - p^{(1}_j(\mathbf{x}-\mathbf{y})p^{2)}_k(\mathbf{x}+\mathbf{y}) + 2\kappa^{(1}_{kj}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) \\ &\quad + 2\kappa^{(1}_{kj}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y}) = J^{1,2}_k J^{1,2}_j + J^{2,1}_k J^{2,1}_j. \end{aligned} \tag{5.18}$$

It is worthwhile to notice here that $Q_{A,j}$ can be expressed using D and J_j , namely

$$Q^{(12)}_{A,j} = \sqrt{D^{1,2}}J^{1,2}_j + \sqrt{D^{2,1}}J^{2,1}_j, \tag{5.19}$$

where $D^{1,2} := \rho^1(\mathbf{x}+\mathbf{y})\rho^2(\mathbf{x}-\mathbf{y})$ and similarly for $D^{2,1}$. Taking into account the observation in (5.18), we can write the tensors $M_{S,kj}$ as a sum of four terms,

$$M^{(12)}_{S,kj} = J^{1,2}_k J^{1,2}_j + J^{2,1}_k J^{2,1}_j + \Phi_{kj} + \delta_{kj}\,[W + P]\,, \tag{5.20}$$

where the tensor Φ_{kj} and the scalar potentials W and P are

$$\Phi_{kj} \stackrel{\text{def}}{=} 2\pi^{(1}_{kj}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + 2\pi^{(1}_{kj}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y}), \tag{5.21a}$$

$$W \stackrel{\text{def}}{=} -[\Delta\rho^{(1}(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + \Delta\rho^{(1}(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y})], \tag{5.21b}$$

$$P \stackrel{\text{def}}{=} 2\big[\rho^{(1}\big]^2(\mathbf{x}+\mathbf{y})\rho^{2)}(\mathbf{x}-\mathbf{y}) + 2\big[\rho^{(1}\big])^2(\mathbf{x}-\mathbf{y})\rho^{2)}(\mathbf{x}+\mathbf{y}). \tag{5.21c}$$

At this point we are ready to derive apriori estimates. For simplicity we will assume that $\psi^1 = \psi^2$, thus $\rho^1 = \rho^2$ etc. in what follows, the assumption makes the formulas look simpler without missing the essential ingredients. The main ingredient will be identity (5.10), which if we contract with a vector field $X^j(\mathbf{y})$ that depends only on the $\mathbf{y}$ variables, we obtain an equation of the form

$$\partial_t \left\{Q_{A,j}X^j\right\} - \nabla_{\mathbf{y},k}\left\{M^k_{S,j}X^j\right\} - \nabla_{\mathbf{x},k}\left\{N^k_{A,j}X^j\right\} + R = 0, \tag{5.22}$$

where the remainder R is a sum of four terms

$$R = R_0 + R_1 + R_2 + R_3 \tag{5.23}$$

given by the following expressions

$$R_0 \stackrel{\text{def}}{=} (\nabla_k X_j)\, J^k J^j; \quad R_1 \stackrel{\text{def}}{=} (\nabla_k X_j)\, \Phi_{kj}, \tag{5.24a}$$

$$R_2 \stackrel{\text{def}}{=} (divX)\, W; \qquad R_3 \stackrel{\text{def}}{=} (divX)\, P. \tag{5.24b}$$

Let us choose the vector field $X^j(\mathbf{y})$ to be of the particular form, we would like to choose

$$X^j(\mathbf{y}) \stackrel{\text{def}}{=} b(|\mathbf{y}|/N)u^j; \quad u^j \stackrel{\text{def}}{=} y^j/|\mathbf{y}|, \tag{5.25}$$

where $b(s)$ is an increasing bounded function and N is a large parameter to be chosen appropriately later. Adopt the convention $r = |\mathbf{y}|$ and compute

$$\nabla_k X_j = \frac{b(r/N)}{r}\left[\delta_{kj} - u_k u_j\right] + \frac{1}{N} b'(r/N) u_k u_j, \tag{5.26a}$$

$$\mathrm{div} X = \frac{1}{r}\left(b(r/N) + (r/N)b'(r/N)\right). \tag{5.26b}$$

Observe that because $b(s)$ is increasing the terms R_0, R_1 and R_3 are positive. We will look more carefully at the R_2 term. First let us examine the three dimensional, case, i.e. $n = 3$ with the simple choice $b = 1$. Let us observe that in three space dimensions we have, see (5.21) for the expression of W

$$R_2 = \int_{R^3\times R^3\times R} \frac{W}{|\mathbf{y}|}\, d\mathbf{x}d\mathbf{y}dt = C\int_{R^3\times R} \rho^1(t,\mathbf{x})\rho^2(t,\mathbf{x})\, d\mathbf{x}dt. \tag{5.27}$$

Recall the notation in (5.1b) and let us define a correlation function $R[T_1, T_2]$ for the density and another correlation, say $L(t)$, between the momentum and the density in the following manner

$$R[T_1,T_2] \stackrel{\text{def}}{=} \int_{D[T_1,T_2]} \rho^1(t,\mathbf{x})\rho^2(t,\mathbf{x})\, d\mathbf{x}dt, \tag{5.28a}$$

$$L(t) \stackrel{\text{def}}{=} \int_{R^n_t\times R^n_t} \left\{Q_{A,j}X^j\right\}\, d\mathbf{x}d\mathbf{y}. \tag{5.28b}$$

With these conventions, after we integrate (5.22), we obtain the estimate

$$R[0,T] \le C\left[L(0) - L(T)\right], \tag{5.29}$$

and one can show, as in [6], that

$$|L(t)| \le C\|\psi^1(t)\|^2_{L^2}\|\psi^2(t)\|^2_{H^{1/2}} + \|\psi^2(t)\|^2_{L^2}\|\psi^1(t)\|^2_{H^{1/2}}. \tag{5.30}$$

It is interesting to recast $L(t)$ in a different light, define a potential function $U^a(t,\mathbf{x})$ and a vector field $P_j^a(t,\mathbf{x})$ by solving Laplace's equation

$$-\Delta U^a = \rho^a; \quad -\Delta P_j^a = p_j^a; \quad a = 1,2, \tag{5.31}$$

now we can write, omiting the indices $a = 1,2$ for simplicity,

$$L(t) = \int_{R_t^3} \left\{ \left(\Delta U P_j - U \Delta P_j\right) x^j \right\} d\mathbf{x}. \tag{5.32}$$

Consider the two dimensional case, i.e. $n = 2$. Assume for simplicity that the two states are equal and make the choice, see (5.25)

$$b(s) \stackrel{\text{def}}{=} s \log(\sqrt{e}/s), \quad \text{if } s \leq 1/\sqrt{e}, \tag{5.33}$$

otherwise we require that $b(s)$ is smooth increasing and bounded by one for $s > 1/\sqrt{e}$. With this choice, we have, with $r = |\mathbf{y}|$

$$\text{div} X = \frac{2}{N} \log(N/r), \quad \text{if } r < N/\sqrt{e}, \tag{5.34}$$

so that the Laplacian of $\text{div}X$ can be written

$$-\Delta\left(\text{div} X\right) = \frac{2}{N}\delta(\mathbf{y}) + \frac{c(r/N)}{N^3}, \tag{5.35}$$

where $c(r/N)$ is a bounded function supported in the region $\{r > N/\sqrt{e}\}$. Integrating (5.22) we obtain the identity below

$$L(T_2) - L(T_1) = \int_{T_1}^{T_2} \int_{R_t^2 \times R_t^2} R \, d\mathbf{x} d\mathbf{y} dt = 0. \tag{5.36}$$

Observe that because X^j is bounded, we can obtain a bound,

$$\int_{R_T^2 \times R_T^2} \left\{ Q_{A,j} X^j \right\} \, d\mathbf{x} d\mathbf{y} \leq \|\psi(T)\|_{L^2}^2 \|\psi(T)\|_{H^{1/2}}^2, \tag{5.37}$$

which estimate follows from the observation

$$\left| \int_{R_t^2} X^j(\mathbf{y}) p_j(t, \mathbf{x}+\mathbf{y}) d\mathbf{x} \right| \leq \left\| X^j \nabla_j \psi \right\|_{H^{-1/2}} \|\psi\|_{H^{1/2}}. \tag{5.38}$$

Let us make the choice $N = (T_2 - T_1)^{\frac{1}{2}}$ so that we have a bound

$$\int_{T_1}^{T_2} \int_{R_t^2 \times R_t^2} \frac{c(|\mathbf{y}|/N)}{N^3} \rho(t, \mathbf{x}+\mathbf{y}) \rho(t, \mathbf{x}-\mathbf{y}) \, d\mathbf{x} d\mathbf{y} dt \leq \|\psi\|_{L^2}^2 / (T_2 - T_1)^{1/2}, \tag{5.39}$$

which follows from the fact that $\int_{R_t^2} \rho(t,\mathbf{x}) \, d\mathbf{x} = \|\psi_0\|_{L^2}^2$. Finally from the integral of the term we called R_2, see (5.24b), we obtain a decay estimate

$$\int_{D[T_1,T_2]} \rho^2(t,\mathbf{x}) \, d\mathbf{x} dt \leq C \sup_t \left\{ \|\psi(t)\|_{L^2}^2 \|\psi(t)\|_{H^{1/2}}^2 \right\} \, (T_1 - T_2)^{\frac{1}{2}}. \tag{5.40}$$

A remark is in order concerning the name correlation for this type of estimate. Let $d(\mathbf{y})$ be the function with the property $\nabla_j d(\mathbf{y}) = X_j(\mathbf{y})$. Multiply (5.12a) with d and differentiate with respect to time, and use (5.10) to obtain the equation

$$\partial_t^2 \{dD\} + \nabla_{\mathbf{y},k} \left\{M_{S,j}^k X^j - d\partial_t Q_A^k\right\} + \nabla_{\mathbf{x},k} \left\{P_{A,j}^k X^j - d\partial_t Q_S^k\right\} = R. \tag{5.41}$$

Integrating with respect to the space variables, we obtain

$$\frac{d^2}{dt^2}\left(\int_{R_t^2\times R_t^2} \{dD\}\ d\mathbf{x}d\mathbf{y}\right) = \int_{R_t^2\times R_t^2} \{R\}\ d\mathbf{x}d\mathbf{y}. \tag{5.42}$$

Notice that the integral

$$C[t] \stackrel{\text{def}}{=} \int_{R_t^2\times R_t^2} \{dD\}\ d\mathbf{x}d\mathbf{y} = \int_{R_t^2\times R_t^2} d(\mathbf{y})\rho^{(1}(t,\mathbf{x}+\mathbf{y})\rho^{2)}(t,\mathbf{x}-\mathbf{y})\ d\mathbf{x}d\mathbf{y}, \tag{5.43}$$

expresses a correlation between the density functions $\rho^a(t,\mathbf{x})$. Since $C[t]$ is positive and its derivative is integrable in the case where the space dimension is three we obtain that $C[t] \sim t$ for large time. The integral $C[t]$ can be written, after a change of variables,

$$C[t] = 2\int_{R_t^2\times R_t^2} |\mathbf{x}-\mathbf{y}|\rho^1(t,\mathbf{x})\rho^2(t,\mathbf{y})d\mathbf{x}d\mathbf{y}.$$

We can derive a true space-time estimate for the two dimensional case. Let $d(t,\mathbf{y})$ and $X^j(t,\mathbf{y})$ to be chosen. The conservation laws that we would like to use are

$$\partial_t \{D\} - \nabla_{\mathbf{y},j} \left\{Q_A^j\right\} - \nabla_{\mathbf{x},j} \left\{Q_S^j\right\} = 0, \tag{5.44a}$$

$$\partial_t \{Q_{A,j}\} - \nabla_{\mathbf{y},k} \left\{M_{S,j}^k\right\} - \nabla_{\mathbf{x},k} \left\{N_{A,j}^k\right\} = 0. \tag{5.44b}$$

Multiply the first equation with $d(t,\mathbf{y})$ and contract the second equation with $X^j(t,\mathbf{y})$ and add them to obtain,

$$\begin{aligned}&\partial_t \left\{dD + X^j Q_{A,j}\right\} - \nabla_{\mathbf{y},k} \left\{dQ_A^k + M_{S,j}^k X^j\right\} - \nabla_{\mathbf{x},k} \left\{dQ_S^k + N_{A,j}^k X^j\right\}\\ &\quad + \left\{-(\partial_t d)D + (\nabla_j d - \partial_t X_j)Q_A^j + (\nabla_k X^j)M_{S,j}^k\right\} = 0.\end{aligned} \tag{5.45}$$

Try the vector field of the form

$$X^j = b\left(\frac{|\mathbf{y}|}{t^\alpha}\right)u^j \tag{5.46}$$

with α to be chosen, so that we have

$$\nabla_k X_j = \frac{b(|\mathbf{y}|/t^\alpha)}{|\mathbf{y}|}[\delta_{kj} - u_k u_j] + \frac{1}{t^\alpha}b'(|\mathbf{y}|/t^\alpha)u_k u_j \tag{5.47a}$$

$$\operatorname{div} X = \frac{1}{|\mathbf{y}|}b(|\mathbf{y}|/t^\alpha) + \frac{1}{t^\alpha}b'(|\mathbf{y}|/t^\alpha). \tag{5.47b}$$

We assumed that $b(s) = s\log\left(\sqrt{e}/s\right)$ for $s \le 1/\sqrt{e}$ and using this we can compute

$$\operatorname{div} X = \frac{2}{t^\alpha}\log\left(t^\alpha/|\mathbf{y}|\right) \quad \text{for } |\mathbf{y}| \le t^\alpha/\sqrt{e}, \tag{5.48}$$

hence we can write

$$\mathrm{div} X = \frac{2}{t^\alpha} \log\big(t^\alpha/|\mathbf{y}|\big) + \frac{1}{t^\alpha} q\big(|\mathbf{y}|/t^\alpha\big), \tag{5.49}$$

where $q(s)$ is a smooth bounded function supported in $s \geq 1/\sqrt{e}$. Let us assume that $d(t, \mathbf{y})$, has the form

$$d(t, \mathbf{y}) = \frac{\alpha}{t^{1-\alpha}} a(|\mathbf{y}|/t^\alpha) \tag{5.50}$$

and choose $a(s)$ such that $a'(s) = sb'(s)$. Combining these we have

$$\nabla_j d - \partial_t X_j = \frac{2\alpha}{t} b'\big(|\mathbf{y}|/t^\alpha\big)\big(|\mathbf{y}|/t^\alpha\big), \tag{5.51}$$

moreover we compute $\partial_t d$ to obtain

$$-\partial_t d = \frac{\alpha(1-\alpha)}{t^{2-\alpha}} a\big(|\mathbf{y}|/t^\alpha\big) + \frac{\alpha^2}{t^{2-\alpha}} a'\big(|\mathbf{y}|/t^\alpha\big)\big(|\mathbf{y}|/t^\alpha\big). \tag{5.52}$$

Finally notice that

$$-\Delta\big(\mathrm{div} X\big) = \frac{2}{t^\alpha}\delta(\mathbf{y}) + \frac{1}{t^{3\alpha}} c\big(|\mathbf{y}|/t^\alpha\big), \tag{5.53}$$

where $c(s)$ is a bounded smooth function supported in $s \geq 1/\sqrt{e}$. For the remainder term in (5.45), we can write

$$\begin{aligned} &-\big(\partial_t d\big) D + \big(\nabla_j d - \partial_t X_j\big) Q_A^j + \big(\nabla_k X^j\big) M_{S,j}^k \\ &\quad = \frac{1}{t^\alpha} b'\big(|\mathbf{y}|/t^\alpha\big) \left[u^j J_j + \frac{\alpha}{t^{1-\alpha}} \frac{|\mathbf{y}|}{t^\alpha} \sqrt{D}\right]^2 + \frac{2}{t^\alpha}\big[\log\big(|\mathbf{y}|/t^\alpha\big) + q\big(|\mathbf{y}|/t^\alpha\big)\big] \frac{W}{|\mathbf{y}|} \\ &\qquad + \text{positive terms.} \end{aligned} \tag{5.54}$$

If we choose $\alpha > 1/3$, then we obtain the estimate

$$\int_{R^2 \times R^+} \frac{\rho^2}{(t+1)^\alpha} d\mathbf{x} dt \leq C \int_{R_0^2 \times R_0^2} \big\{dD + X^j Q_{A,j}\big\} \; d\mathbf{x} d\mathbf{y} - L(\infty), \tag{5.55}$$

where

$$L(\infty) = \lim_{t \to +\infty} \int_{R_t^2 \times R_t^2} \big\{Q_{A,j} X^j\big\} \; d\mathbf{x} d\mathbf{y}, \tag{5.56}$$

which estimate expresses a weak type of decay for the square density.

An interesting observation is the following, by considering two solutions $\psi(t, \mathbf{x})$ and $\psi(t + t_0, \mathbf{x} + \mathbf{x}_0)$, we can derive estimates for the integral

$$\int_{R^n \times R} \rho(t, \mathbf{x}) \rho(t + t_0, \mathbf{x} + \mathbf{x}_0) \; d\mathbf{x} dt.$$

Another idea is to employ the pseudo-conformal conservation. We know that if $\psi(t, \mathbf{x})$ is a solution, then

$$S(t, \mathbf{x}) \psi\left(-1/t, \mathbf{x}/t\right), \quad \text{where } S(t, \mathbf{x}) = \frac{1}{t} e^{i|\mathbf{x}|^2/4t}$$

is also a solution so we can estimate the integral

$$\int_{R^2 \times R^+} \frac{1}{t^2} \rho(t, \mathbf{x}) \rho\left(-1/t, \mathbf{x}/t\right) \; d\mathbf{x} dt.$$

References

[1] J. Bourgain, *Global Solution of Nonlinear Schrödinger Equations*, American Mathematical Society Colloquium Publications, Vol. 46 (1999).
[2] J. Bourgain, Refinements of Strichartzs' inequality and applications to 2D-NLS with critical nonlinearity, *Int. Math. Res. Notices* (1998).
[3] R. Coifman and Y. Meyer, *Au dela des Operateurs Pseudo-Differentiels*, Asterisque, Vol. 57 (Societe Mathematique de France, Paris, 1978).
[4] R. Coifman and Y. Meyer, Commutateurs d'integrales singulieres et operateurs multilineaires, *Ann. Inst. Fourier (Grenoble)* **28** (1978) 177–202.
[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Almost conservation laws and global rough solutions to a nonlinear Schrodinger equation, preprint.
[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global existence and scattering for rough solutions of a nonlinear Schrodinger equation on R^3, preprint.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Generalizations of the interaction morawetz inequality, preprint.
[8] J. Colliander, J. Delort, C. Kenig and G. Staffilani, Bilinear estimates and applications to 2D NLS, *Trans. Amer. Math. Soc.* **353**(8) (2001) 3307–3325.
[9] J. Lin and W. Strauss, Decay and scattering of solutions of a nonlinear Schrodinger equation, *J. Funct. Anal.* **30** (1978) 245–263.
[10] T. Tao, Multilinear weighted convolution of L^2 functions and applications to nonlinear dispersive equations, *Amer. J. Math.* **123** (2001) 839–908.