1. Complex Logarithm

Theorem 1.1. Let Ω be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F(z)$ on Ω such that

(1) $F(1) = 0$ and $F'(z) = 1/z$ on Ω.
(2) $e^{F(z)} = z$ for all $z \in \Omega$
(3) $F(r) = \ln r$ when r is a positive real number close to 1.

Proof. Since $0 \notin \Omega$, $1/z$ is holomorphic on Ω. Let γ be a piecewise smooth curve connecting 1 and z. Define

$$F(z) = \int_\gamma \frac{dw}{w}.$$

Since Ω is simply connected and $1/z$ is holomorphic on Ω, $F(z)$ is independent of choice of γ. Hence we write

$$F(z) = \int_1^z \frac{dw}{w}, \quad z \in \Omega.$$

Then F defines a function on Ω with $F(1) = 0$. It is easy to verify that $F(z)$ is holomorphic such that $F'(z) = 1/z$ on Ω. To prove (2), we let

$$g(z) = ze^{-F(z)}, \quad z \in \Omega.$$

Then $g(z)$ is holomorphic on Ω with $g(1) = 1$. Then

$$\begin{aligned} g'(z) &= e^{-F(z)} + z \cdot e^{-F(z)} \cdot (-F'(z)) \\ &= e^{-F(z)} - z \cdot e^{-F(z)} \cdot \frac{1}{z} \\ &= e^{-F(z)} - e^{-F(z)} = 0. \end{aligned}$$

Since Ω is a simply connected domain and $g'(z) = 0$ on Ω, $g(z)$ is a constant function. Thus $g(z) = g(1) = 1$ for all $z \in \Omega$.

When r is real and close to 1, then

$$F(r) = \int_1^r \frac{dw}{w}.$$

Let us choose the path from 1 to r on the axis defined by $w(t) = t$ for $1 \le t \le r$ if $r \ge 1$, or $r \le t \le 1$ if $r < 1$. Then

$$F(t) = \int_1^r \frac{dt}{t} = \ln r.$$

Notice that such a function is unique by the coincidence principle. □

We see that $F(z)$ is an extension of the standard logarithm. We call $F(z)$ a branch of $\log z$ and denote $F(z)$ by $\log_\Omega z$. For example, $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is a simply connected domain in $\mathbb{C}$ such that $0 \notin \Omega$ and $1 \in \Omega$. Then there is a unique holomorphic function F on Ω satisfying (1)-(3). This function is defined by the complex integration:

$$F(z) = \int_1^z \frac{dw}{w}.$$

Suppose $z = re^\theta$ for $-\pi < \theta < \pi$. Without loss of generality, we assume $r > 1$. Let L_1 be the path from 1 to r defined by $w(t) = t$ for $1 \le t \le r$ and L_2 be the arc $w(t) = re^{it}$ for

$0 \le t \le \theta$. Then the path integral is equal to

$$\begin{aligned} F(z) &= \int_{L_1} \frac{dw}{w} + \int_{L_2} \frac{dw}{w} \\ &= \int_1^r \frac{dt}{t} + \int_0^\theta \frac{rie^{it}}{e^{it}} dt \\ &= \ln r + i\theta. \end{aligned}$$

Corollary 1.1. Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and $F(z)$ be the branched of $\log z$ in Ω. If $z = re^{i\theta}$ for $r > 0$ and $-\pi < \theta < \pi$, then

$$F(z) = \ln r + i\theta.$$

Definition 1.1. The branch of $\log z$ defined on Ω is called the principal branch of $\log z$.

In calculus, we have already learned that $\ln(1+x)$ has the following series expansion:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}, \quad -1 < x < 1.$$

Let $F(z)$ be the principal branch of $\log z$. We will prove that

$$F(1+z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}, \quad |z| < 1.$$

Let $G(z)$ be the power series $\sum_{n=1}^{\infty} (-1)^n z^n / n$ defined on $\mathbb{D} = \{z : |z| < 1\}$. Then $G(z)$ is holomorphic on $\mathbb{D}$ For any $0 < r < 1$, we have

$$F(1+r) = \ln(1+r) = G(r).$$

We find $F(1+z) = G(z)$ on $[0,1]$. Since $\mathbb{D} \setminus (-\infty, 0]$ is connected and $F(1+z) = G(z)$ on $[0,1]$, by coincidence principle, $F(1+z) = G(z)$ for all $\mathbb{D} \setminus (-\infty, 0]$.

Theorem 1.2. Let f be a nowhere vanishing holomorphic function on a simply connected domain Ω. Then there exists a holomorphic function g on Ω such that

$$f(z) = e^{g(z)}.$$

In this case, we denote $g(z)$ by $\log f(z)$. It determine a branch of $\log z$.

Proof. Given $z_0 \in \Omega$, let c_0 be a constant such that $e^{c_0} = f(z_0)$. Define

$$g(z) = c_0 + \int_{z_0}^{z} \frac{f'(w)}{f(w)} dw.$$

Since f is nonzero on Ω, f'/f is holomorphic on Ω. Since Ω is simply connected, the path integral is independent of choice of paths from z_0 to z. Moreover, g is holomorphic on Ω such that $g'(z) = f'(z)/f(z)$. Define a holomorphic function $G(z)$ on Ω by

$$G(z) = f(z)e^{-g(z)}.$$

Then $G'(z) = 0$ on Ω. Since Ω is simply connected, G is a constant function. Since $G(z_0) = f(z_0)e^{-c_0} = 1$, we see that $G(z) = 1$ on Ω. Thus $f(z) = e^{g(z)}$ on Ω. $\square$

Let $y_0 \in \mathbb{R}$ and denote

$$A_{y_0} = \{x + iy : x \in \mathbb{R},\ y_0 \le y < y_0 + 2\pi\}.$$

Then one can show that the function

$$\exp : A_{y_0} \to \mathbb{C} \setminus \{0\}$$

is a bijection. In other word, we may define $\log_{A_{y_0}}$ to be the inverse of $\exp : A_{y_0} \to \mathbb{C} \setminus \{0\}$. Thus $\log_{A_{y_0}}$ is a branch of $\log z$. When $y_0 = -\pi$, we obtain the (extension of) principal branch of $\log z$.

Proposition 1.1. Choose a branch of $\log z$. For any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\log(z_1 \cdot z_2) = \log z_1 + \log z_2 \mod 2\pi.$$

After choosing a branch of $\log z$, we define

$$z^\alpha = e^{\alpha \log z}$$

for any $\alpha \in \mathbb{C}$. The n-th root function $\sqrt[n]{z}$ is defined by

$$\sqrt[n]{z} = z^{\frac{1}{n}}$$

when a branch of $\log z$ is chosen. In this case, we call $\sqrt[n]{z}$ is a branch of the n-th root function.

Proposition 1.2. Let $-\pi < \theta < \pi$ and $z = re^{i\theta}$. Choose the principal branch for $\log z$. Then

$$\sqrt[n]{z} = \sqrt{r} e^{\frac{i\theta}{n}}.$$

Proof. Let $\log z$ be the principal branch of $\log$. Then $\log z = r + i\theta$. Thus

$$\sqrt[n]{z} = e^{\frac{1}{n}\log z} = e^{\frac{\ln r + i\theta}{n}} = e^{\frac{\ln r}{n}} e^{\frac{\theta}{n} i} = \sqrt[n]{r} e^{i\frac{\theta}{n}}.$$

$\square$