1. **Complex Logarithm**

**Theorem 1.1.** Let $\Omega$ be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F(z)$ on $\Omega$ such that

1. $F(1) = 0$ and $F'(z) = 1/z$ on $\Omega$.
2. $e^{F(z)} = z$ for all $z \in \Omega$.
3. $F(r) = \ln r$ when $r$ is a positive real number close to 1.

**Proof.** Since $0 \notin \Omega$, $1/z$ is holomorphic on $\Omega$. Let $\gamma$ be a piecewise smooth curve connecting $1$ and $z$. Define

$$F(z) = \int_{\gamma} \frac{dw}{w}.$$ 

Since $\Omega$ is simply connected and $1/z$ is holomorphic on $\Omega$, $F(z)$ is independent of choice of $\gamma$. Hence we write

$$F(z) = \int_{1}^{z} \frac{dw}{w}, \quad z \in \Omega.$$ 

Then $F$ defines a function on $\Omega$ with $F(1) = 0$. It is easy to verify that $F(z)$ is holomorphic such that $F'(z) = 1/z$ on $\Omega$. To prove (2), we let $g(z) = ze^{-F(z)}$, $z \in \Omega$.

Then $g(z)$ is holomorphic on $\Omega$ with $g(1) = 1$. Then

$$g'(z) = e^{-F(z)} + z \cdot e^{-F(z)} \cdot (-F'(z))$$

$$= e^{-F(z)} - z \cdot e^{-F(z)} \cdot \frac{1}{z}$$

$$= e^{-F(z)} - e^{-F(z)} = 0.$$ 

Since $\Omega$ is a simply connected domain and $g'(z) = 0$ on $\Omega$, $g(z)$ is a constant function. Thus $g(z) = g(1) = 1$ for all $z \in \Omega$.

When $r$ is real and close to 1, then

$$F(r) = \int_{1}^{r} \frac{dw}{w}.$$ 

Let us choose the path from 1 to $r$ on the axis defined by $w(t) = t$ for $1 \leq t \leq r$ if $r \geq 1$, or $r \leq t \leq 1$ if $r < 1$. Then

$$F(t) = \int_{1}^{r} \frac{dt}{t} = \ln r.$$ 

Notice that such a function is unique by the coincidence principle. \qed

We see that $F(z)$ is an extension of the standard logarithm. We call $F(z)$ a branch of $\log z$ and denote $F(z)$ by $\log_{\Omega} z$. For example, $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is a simply connected domain in $\mathbb{C}$ such that $0 \notin \Omega$ and $1 \in \Omega$. Then there is a unique holomorphic function $F$ on $\Omega$ satisfying (1)-(3). This function is defined by the complex integration:

$$F(z) = \int_{1}^{z} \frac{dw}{w}.$$ 

Suppose $z = re^{i\theta}$ for $-\pi < \theta < \pi$. Without loss of generality, we assume $r > 1$. Let $L_{1}$ be the path from 1 to $r$ defined by $w(t) = t$ for $1 \leq t \leq r$ and $L_{2}$ be the arc $w(t) = re^{it}$ for $0 \leq t < 1$. Then

$$F(z) = \int_{L_{1}} \frac{dw}{w} + \int_{L_{2}} \frac{dw}{w}.$$ 

Notice that $F(z)$ is well-defined and holomorphic on $\Omega$. \qed
Then one can show that the function
\[ f(z) = \int_{L_1} \frac{dw}{w} + \int_{L_2} \frac{dw}{w} \]
\[ = \int_1^r \frac{dt}{t} + \int_0^\theta r e^{it} dt \]
\[ = \ln r + i\theta. \]

**Corollary 1.1.** Let \( \Omega = \mathbb{C} \setminus (-\infty, 0] \) and \( F(z) \) be the branched of log \( z \) in \( \Omega \). If \( z = re^{i\theta} \) for \( r > 0 \) and \( -\pi < \theta < \pi \), then
\[ F(z) = \ln r + i\theta. \]

**Definition 1.1.** The branch of log \( z \) defined on \( \Omega \) is called the principal branch of log \( z \). In calculus, we have already learned that \( \ln(1 + x) \) has the following series expansion:
\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^\infty (-1)^{n+1} \frac{x^n}{n}, \quad -1 < x < 1. \]
Let \( F(z) \) be the principal branch of log \( z \). We will prove that
\[ F(1 + z) = \sum_{n=1}^\infty (-1)^n \frac{z^n}{n}, \quad |z| < 1. \]
Let \( G(z) \) be the power series \( \sum_{n=1}^\infty (-1)^n \frac{z^n}{n} \) defined on \( \mathbb{D} = \{ z : |z| < 1 \} \). Then \( G(z) \) is holomorphic on \( \mathbb{D} \) For any \( 0 < r < 1 \), we have
\[ F(1 + r) = \ln(1 + r) = G(r). \]
We find \( F(1 + z) = G(z) \) on \( [0, 1] \). Since \( \mathbb{D} \setminus (-\infty, 0] \) is connected and \( F(1 + z) = G(z) \) on \( [0, 1] \), by coincidence principle, \( F(1 + z) = G(z) \) for all \( \mathbb{D} \setminus (-\infty, 0] \).

**Theorem 1.2.** Let \( f \) be a nowhere vanishing holomorphic function on a simply connected domain \( \Omega \). Then there exists a holomorphic function \( g \) on \( \Omega \) such that
\[ f(z) = e^{g(z)}. \]
In this case, we denote \( g(z) \) by \( \log f(z) \). It determine a branch of log \( z \).

**Proof.** Given \( z_0 \in \Omega \), let \( c_0 \) be a constant such that \( e^{c_0} = f(z_0) \). Define
\[ g(z) = c_0 + \int_{z_0}^z \frac{f'(w)}{f(w)} \, dw. \]
Since \( f \) is nonzero on \( \Omega \), \( f'/f \) is holomorphic on \( \Omega \). Since \( \Omega \) is simply connected, the path integral is independent of choice of paths from \( z_0 \) to \( z \). Moreover, \( g \) is holomorphic on \( \Omega \) such that \( g'(z) = f'(z)/f(z) \). Define a holomorphic function \( G(z) \) on \( \Omega \) by
\[ G(z) = f(z)e^{-g(z)}. \]
Then \( G'(z) = 0 \) on \( \Omega \). Since \( \Omega \) is simply connected, \( G \) is a constant function. Since \( G(z_0) = f(z_0)e^{-c_0} = 1 \), we see that \( G(z) = 1 \) on \( \Omega \). Thus \( f(z) = e^{g(z)} \) on \( \Omega \).

Let \( y_0 \in \mathbb{R} \) and denote
\[ A_{y_0} = \{ x + iy : x \in \mathbb{R}, \ y_0 \leq y < y_0 + 2\pi \}. \]
Then one can show that the function
\[ \exp : A_{y_0} \to \mathbb{C} \setminus \{0\} \]
is a bijection. In other word, we may define \( \log_{A_{y_0}} \) to be the inverse of \( \exp : A_{y_0} \to \mathbb{C} \setminus \{0\} \). Thus \( \log_{A_{y_0}} \) is a branch of \( \log z \). When \( y_0 = -\pi \), we obtain the (extension of) principal branch of \( \log z \).

**Proposition 1.1.** Choose a branch of \( \log z \). For any \( z_1, z_2 \in \mathbb{C} \setminus \{0\} \),

\[
\log(z_1 \cdot z_2) = \log z_1 + \log z_2 \mod 2\pi.
\]

After choosing a branch of \( \log z \), we define

\[
z^{\alpha} = e^{\alpha \log z}
\]

for any \( \alpha \in \mathbb{C} \). The \( n \)-th root function \( n^{\sqrt{z}} \) is defined by

\[
n^{\sqrt{z}} = z^{1/n}
\]

when a branch of \( \log z \) is chosen. In this case, we call \( n^{\sqrt{z}} \) is a branch of the \( n \)-th root function.

**Proposition 1.2.** Let \(-\pi < \theta < \pi \) and \( z = re^{i\theta} \). Choose the principal branch for \( \log z \). Then

\[
n^{\sqrt{z}} = \sqrt{re^{i\theta}}.
\]

**Proof.** Let \( \log z \) be the principal branch of \( \log \). Then \( \log z = r + i\theta \). Thus

\[
n^{\sqrt{z}} = \sqrt[4]{n} \log z = e^{\frac{\ln r + i\theta}{4}} = e^{\frac{\ln r}{4}} e^{rac{i\theta}{4}} = \sqrt[4]{r} e^{\frac{i\theta}{4}}.
\]

\[\square\]