1. Space of Harmonic Polynomials

Let $\mathbb{R}[x, y]$ be the space of polynomials in $x, y$ over $\mathbb{R}$. The differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

defines a linear operator on $\mathbb{R}[x, y]$. Elements of $V = \ker \Delta$ are called harmonic polynomials, i.e. $P \in V$ if and only if $\Delta P = 0$.

A polynomial $P(x, y)$ is called homogeneous of degree $n$ if

$$P(\lambda x, \lambda y) = \lambda^n P(x, y)$$

for all $\lambda > 0$. Let $V_n$ be the space of homogeneous harmonic polynomials of degree $n$. Then it forms a vector subspace of $V$. Let $P$ be a homogeneous polynomial of degree $n$. Then

$$P(r \cos \theta, r \sin \theta) = r^n P(\cos \theta, \sin \theta).$$

Denote $P(\cos \theta, \sin \theta) = \Phi(\theta)$. Using polar coordinate for

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

we see that $\Delta P = 0$ if and only if

$$\Phi''(\theta) + n^2 \Phi(\theta) = 0.$$ 

By solving this O.D.E, we obtain $\Phi(\theta) = a \cos n\theta + b \sin n\theta$. Hence

$$P(x, y) = r^n(a \cos n\theta + b \sin n\theta) = aC_n(x, y) + bS_n(x, y).$$

Here $C_n(x, y) = \text{Re}(x + iy)^n$ and $S_n(x, y) = \text{Im}(x + iy)^n$. This shows that $V_n$ is spanned by $\{C_n(x, y), S_n(x, y)\}$. Moreover, it is easy to see $\{C_n(x, y), S_n(x, y)\}$ is linearly independent (you can use Polar coordinate again. Hence we simply use the fact that $\{\cos n\theta, \sin n\theta\}$ is linearly independent.) In other words, $\{C_n(x, y), S_n(x, y)\}$ forms a basis for $V_n$.

Notice that every polynomial can be written as a sum of homogeneous polynomials. In fact, given a polynomial $f(x, y) = \sum_{i+j=n} a_{ij} x^i y^j$ of degree $N$, we set $f_n(x, y) = \sum_{i+j=n} a_{ij} x^i y^j$;

then $f(x, y) = f_0(x, y) + f_1(x, y) + \cdots + f_N(x, y)$. Moreover $f_n(x, y)$ is homogeneous of degree $n$. If $\Delta f = 0$, then

$$\Delta f_0 + \Delta f_1 + \cdots + \Delta f_N = 0.$$ 

Since constants and degree one polynomials are already harmonic, we see that

$$\Delta f_2 + \Delta f_3 + \cdots + \Delta f_N = 0.$$ 

Since $\Delta f_i$ are homogeneous of different degree, $\{\Delta f_i : 2 \leq i \leq N\}$ is linearly independent. Hence $\Delta f_i(x, y) = 0$ for $0 \leq i \leq N$. In other words, for any $f \in V$, $f \in \bigoplus_{n=0}^{\infty} V_n$. It is obvious that $\bigoplus_{n=0}^{\infty} V_n$ is a subspace of $V$. Hence we obtain

$$V = \bigoplus_{n=0}^{\infty} V_n.$$