1. Space of Bounded Functions and Space of Continuous functions

Let X be a nonempty set. A real-valued function $f : X \to \mathbb{R}$ is bounded if there exists $M > 0$ so that $|f(x)| \le M$, for all $x \in X$. The set of bounded real-valued functions on X is denoted by $\mathcal{B}(X)$. Given $f, g \in \mathcal{B}(X)$, and $a \in \mathbb{R}$, we define

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x)$$

for $x \in X$.

Proposition 1.1. The set $\mathcal{B}(X)$ forms a real vector space.

Proof. We leave it to the reader as an exercise. $\square$

Let V be a real vector space. A norm on V is a function $\|\cdot\| : V \to [0,\infty)$ such that

(1) $\|av\| = |a|\|v\|$ for all $a \in \mathbb{R}$ and $v \in V$.
(2) $\|v\| = 0$ if and only if $v = 0$.
(3) $\|v + w\| \le \|v\| + \|w\|$ for $v, w \in V$.

A normed vector space over $\mathbb{R}$ is a real vector space together with a norm. It is easy to see that a norm on V induces a metric on V by

$$d(v, w) = \|v - w\|.$$

The metric defined above is called the metric induced from the norm.

Definition 1.1. A normed vector space over $\mathbb{R}$ is called a real Banach space if the space V together with the metric induced from the norm is complete.

On $\mathcal{B}(X)$, set

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Definition 1.2. If a sequence $\{f_n\}$ converges to f in $\mathcal{B}(X)$, we say that $\{f_n\}$ converges uniformly to f on X.

If $\{f_n\}$ converges to f in $\mathcal{B}(X)$, by definition, given $\epsilon > 0$, there exists $N > 0$ so that for all $n \ge N$, $\|f_n - f\|_\infty < \epsilon$. Hence for all $x \in X$, and $n \ge N$, $|f_n(x) - f(x)| \le \|f_n - f\|_\infty < \epsilon$. In other words, $\{f_n\}$ converges uniformly to f if given $\epsilon > 0$, there exists $N > 0$ so that for all $n \ge N$, and all $x \in X$, $|f_n(x) - f(x)| < \epsilon$.

Proposition 1.2. The normed space $(\mathcal{B}(X), \|\cdot\|_\infty)$ is a real Banach space.

Proof. Denote $V = \mathcal{B}(X)$. To show that V is complete, we need to show that every Cauchy sequence is convergent.

Let $\{f_n\}$ be a Cauchy sequence in V. Given $\epsilon > 0$, there exists $N > 0$ such that $\|f_n - f_m\|_\infty < \epsilon$. Hence for all $x \in X$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \epsilon.$$

This implies that for each $x \in X$, the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, $\{f_n(x)\}$ is convergent. Let the limit of $\{f_n(x)\}$ be $f(x)$, i.e. $f(x) = \lim_{n\to\infty} f_n(x)$. Hence we obtain a function $f(x)$ on X. On the other hand, a Cauchy sequence in a normed space must be bounded. There is $M > 0$ so that $\|f_n\|_\infty \le M$. In other words, for each $x \in X$, $|f_n(x)| \le \|f_n\|_\infty \le M$. Taking $n \to \infty$, we find $|f(x)| = \lim_{n\to\infty} |f_n(x)| \le M$, for each $x \in X$. This shows that $f(x)$ is a bounded function on X and thus $f \in V$. For each $x \in X$, and $n \ge N$,

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \epsilon.$$

This shows that for $n \geq N$, $\|f_n - f\|_\infty \leq \epsilon$. We find f is the limit of $\{f_n\}$ in V. We prove that $\{f_n\}$ is convergent in V. $\square$

If (X, d) is a compact metric space, we can talk more about $\mathcal{B}(X)$. Let $C(X)$ be the space of all real-valued continuous functions on X. Since X is compact, every continuous function on X is bounded. Therefore $C(X)$ is a subset of $\mathcal{B}(X)$. Moreover, since the sum of continuous functions on X is continuous function on X and the scalar multiplication of a continuous function by a real number is again continuous, it is easy to check that $C(X)$ is a vector subspace of $\mathcal{B}(X)$.

Definition 1.3. Let (M, d) be a metric space and A be a subset of M. We say that $a \in M$ is a limit point of A if there exists a sequence $\{a_n\}$ of elements of A whose limit is a. A is said to be closed if A contains all of its limit points.

Proposition 1.3. Let (X, d) be a compact metric space. The space $C(X)$ of real-valued continuous functions is a closed subset of the space $\mathcal{B}(X)$ of bounded real-valued functions on X.

Proof. To show that $C(X)$ is closed in $\mathcal{B}(X)$, we only need to show that $C(X)$ contains all of its limit points.

Let f be a bounded real-valued function so that f is a limit point of $C(X)$. There exists $\{f_n\}$ in $C(X)$ so that $\{f_n\}$ converges to f in $\mathcal{B}(X)$. To show $f \in C(X)$, we need to show that f is a continuous function.

Given $\epsilon > 0$, we can choose $N > 0$ so that $\|f_N - f\|_\infty < \epsilon/3$. Since f_N is uniformly continuous[1], there exists $\delta > 0$, so that if $d(x, y) < \delta$, $|f_N(x) - f_N(y)| < \epsilon/3$. If $d(x, y) < \delta$, we see

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq 2\|f - f_N\|_\infty + \frac{\epsilon}{3} < \epsilon.$$

This shows that f is uniformly continuous on X and hence continuous. $\square$

Remark. It is equivalent to say that the uniform limit of a sequence of continuous functions is again continuous.

Let (M, d) be a metric space and N be a subset of M. On N, we set

$$d_N(x, y) = d(x, y), \quad x, y \in N.$$

Then (N, d_N) is again a metric space. We call (N, d_N) the metric subspace of (M, d) and d_N the metric induced from d.

Proposition 1.4. Let N be a closed subset of a complete metric space (M, d). Then (N, d_N) is also a complete metric space.

Proof. To show that N is complete, we show that every Cauchy sequence in N has a limit in N.

Let $\{a_n\}$ be a Cauchy sequence in N. Then $\{a_n\}$ is a Cauchy sequence in M. Since M is complete, $\{a_n\}$ is convergent to a point $a \in M$. This implies that a is a limit point of N. Since N is closed, $a \in N$. Hence $\{a_n\}$ has a limit a in N.

$\square$

Corollary 1.1. The space $C(X)$ is a real Banach space.

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[1] A continuous function on a compact space is uniformly continuous

Proof. By Proposition 1.2, $\mathcal{B}(X)$ is complete. By Proposition 1.3, $C(X)$ is a closed subset of $\mathcal{B}(X)$. By Proposition 1.4, $C(X)$ is complete.

$\square$