1. Completion of a metric space

A metric space need not be complete. For example, let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open ball in $\mathbb{R}^2$. The metric subspace $(B, d_B)$ of $\mathbb{R}^2$ is not a complete metric space.

**Proposition 1.1.** Let $(X, d_X)$ be a complete metric space and $Y$ be a subset of $X$. Then $(Y, d_Y)$ is complete if and only if $Y$ is a closed subset of $X$.

*Proof.* This is left to the reader as an exercise. \hfill \Box

This proposition allows us to construct many examples of metric spaces which are not complete.

Let $\mathbb{D} = \overline{B}$ be the closure of $B$ in $\mathbb{R}^2$. Proposition 1.1 implies that $(\mathbb{D}, d_{\mathbb{D}})$ is a complete metric space. Although $(B, d_B)$ is not complete, $B$ is a “dense” subset of a complete metric space $(\mathbb{D}, d_{\mathbb{D}})$.

**Definition 1.1.** Let $(M, d)$ be a metric space and $D$ be a subset of $M$. We say that $D$ is a dense subset of $M$ if $\overline{D} = M$.

**Example 1.1.** The set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

These observations lead to the notion of completion of a metric space.

**Definition 1.2.** Let $(X, d)$ be a metric space. A completion of $(X, d)$ is a complete metric space $(\hat{X}, \hat{d})$ together with a function $j : X \to \hat{X}$ such that

$$\hat{d}(j(x), j(y)) = d(x, y), \quad \text{for any } x, y \in X$$

and $j(X)$ is dense in $\hat{X}$, i.e. $\overline{j(X)} = \hat{X}$.

A function $f : (X, d) \to (Y, \rho)$ is called an isometry if $\rho(f(x), f(x')) = d(x, x')$ for any $x, x' \in X$. An isometry is always injective.

**Definition 1.3.** A function $f : (X, d) \to (Y, \rho)$ is an isomorphism if $f$ is an isometry and $f$ is surjective. Two metric spaces are called isomorphic if there is an isomorphism between them.

**Remark.** If $f : X \to Y$ is an isometry, then $X$ is always isomorphic to $f(X)$.

Recall that $B(S)$ is a Banach space. Denote $d_\infty(f, g) = \|f - g\|_\infty$.

**Theorem 1.1.** Any metric space $(X, d)$ is isomorphic to a metric subspace of $(B(X), d_\infty)$.

*Proof.* Let $(X, d)$ be a metric space. For each $x \in X$, we define a function $f_x : X \to \mathbb{R}$ by $f_x(t) = d(x, t)$. Let us fixed $x_0 \in X$. Observe that

$$|f_x(t) - f_{x_0}(t)| = |d(x, t) - d(x_0, t)| \leq d(x, x_0) \text{ for any } t \in X.$$ 

Hence if we define another function $j_x : X \to \mathbb{R}$ by $j_x = f_x - f_{x_0}$. Then $j_x$ is a real valued bounded function on $X$, i.e. $j_x \in B(X)$. Moreover,

$$|j_x(t) - j_y(t)| = |(f_x(t) - f_{x_0}(t)) - (f_y(t) - f_{y_0}(t))| = |f_x(t) - f_y(t)| \leq d(x, y)$$

for any $t \in X$. Hence $\|j_x - j_y\|_\infty \leq d(x, y)$. In fact,

$$|j_x(y) - j_y(y)| = |f_x(y) - f_y(y)| = |d(x, y)| = d(x, y).$$

We see that $\|j_x - j_y\|_\infty = d(x, y)$, i.e. $d_\infty(j_x, j_y) = d(x, y)$ for any $x, y \in X$. We obtain an isometry

$$j : X \to B(X), \quad x \mapsto j_x.$$ 

Let $j(X) = Y$ and $(Y, d_Y)$ be the metric subspace of $(B(X), d_\infty)$ associated with $Y$. Since $j$ is an isometry and $j(X) = Y$, $j : X \to Y$ is an isomorphism of metric spaces. \hfill \Box
Corollary 1.1. Every metric space has a completion.

Proof. Let \( j : X \to B(X) \) as above and \( \hat{X} = j(X) \) and \( (\hat{X}, \hat{d}) \) be the associated metric subspace of \((B(X), d_\infty)\). Since \( \hat{X} \) is closed in \( B(X) \) and \( B(X) \) is complete, \((\hat{X}, \hat{d})\) is complete by Proposition 1.1. We obtain a completion of \((X, d)\). \( \square \)

Theorem 1.2. Let \((X, d)\) be a metric space. Suppose \((X_1, d_1, j_1)\) and \((X_2, d_2, j_2)\) are two completions of \((X, d)\). Then \((X_1, d_1)\) and \((X_2, d_2)\) are isomorphic.

Before proving this theorem, we need the following Lemma.

Lemma 1.1. Let \((X, d)\) and \((Y, \rho)\) be metric spaces and \(D\) be a dense subset of \(X\). Suppose \(Y\) is complete and \(f : D \to Y\) is an isometry. Then there exists a unique isometry \(F : X \to Y\) so that \(F|_D = f\).

Proof. When \(x \in D\), we define \(F(x) = f(x)\). Now, we want to define \(F(x)\) for \(x \in X \setminus D\). Since \(D\) is dense in \(X\), for any \(x \in X\), we can choose a sequence \((x_n)\) in \(D\) so that \((x_n)\) is convergent to \(x\) in \(X\). Since \((x_n)\) is convergent to \(x\) in \(X\), it is a Cauchy sequence in \(X\). For any \(\epsilon > 0\), there exists \(N_\epsilon \in \mathbb{N}\) so that \(d(x_n, x_m) < \epsilon\) whenever \(n, m \geq N_\epsilon\). Since \(f\) is an isometry,

\[
\rho(f(x_n), f(x_m)) = d(x_n, x_m) < \epsilon
\]

whenever \(n, m \geq N_\epsilon\). This shows that \((f(x_n))\) is a Cauchy sequence in \(Y\). Since \(Y\) is complete, \((f(x_n))\) is convergent. Denote the limit of \((f(x_n))\) by \(y\). We may define \(F(x)\) by \(y\).

We need to make sure that \(y\) is uniquely determined by \(x\). Let \((x'_n)\) be another sequence in \(X\) which converges to \(x\). Claim \((f(x'_n))\) is also convergent to \(y\). Let \(y'\) be the limit of \((f(x'_n))\). By triangle inequality,

\[
\rho(y, y') \leq \rho(y, f(x_n)) + \rho(f(x_n), f(x'_n)) + \rho(f(x'_n), y')
\]

\[
\leq \rho(y, f(x_n)) + d(x_n, x'_n) + \rho(f(x'_n), y')
\]

\[
\leq \rho(y, f(x_n)) + d(x_n, x) + d(x'_n, x) + \rho(f(x'_n), y').
\]

Since \((x_n)\) and \((x'_n)\) are convergent to \(x\) and \((f(x_n))\) is convergent to \(y\) and \((f(x'_n))\) is convergent to \(y'\), for any \(\epsilon > 0\), we choose \(N_\epsilon \in \mathbb{N}\) so that \(d(x_n, x)\) and \(d(x'_n, x)\) and \(\rho(f(x_n), y)\) and \(\rho(f(x'_n), y')\) are all less than \(\epsilon/4\). This implies that \(\rho(y, y') < \epsilon\) for any \(\epsilon > 0\). This implies \(y = y'\). Thus we obtain a function \(F : X \to Y\) defined by

\[
F(x) = \begin{cases} 
  f(x) & \text{if } x \in D \\
  \lim_{n \to \infty} f(x_n) & \text{if } x \in X \setminus D \text{ and } x = \lim_{n \to \infty} x_n.
\end{cases}
\]

Let us prove that \(F\) is an isometry. When \(x, y \in D\), \(F(x) = f(x)\) and \(F(y) = f(y)\). In this case, \(\rho(F(x), F(y)) = \rho(f(x), f(y)) = d(x, y)\). Let us prove the case when \(x \in X \setminus D\) and \(y \in D\). Then \(\rho(F(x), F(y)) = \rho(F(x), f(y))\). We want to show \(\rho(F(x), f(y)) = d(x, y)\). Choose a sequence \((x_n)\) which converges to \(x\) in \(X\). By triangle inequality,

\[
|d(x_n, y) - d(x, y)| \leq d(x_n, x)
\]

we find \(\lim_{n \to \infty} d(x_n, x) = d(x, y)\). By triangle inequality,

\[
|\rho(F(x), f(y)) - \rho(f(x_n), f(y))| \leq d(f(x_n), F(x)),
\]

we find \(\lim_{n \to \infty} \rho(f(x_n), f(y)) = \rho(F(x), f(y))\). Since \(x_n, y \in D\), \(\rho(f(x_n), f(y)) = d(x_n, y)\). This implies that

\[
\rho(F(x), f(y)) = \lim_{n \to \infty} \rho(f(x_n), f(y)) = \lim_{n \to \infty} d(x_n, y) = d(x, y).
\]
Finally, let us show that for \( x, y \in X \setminus D \), \( \rho(F(x), F(y)) = d(x, y) \). Choose a sequence \( (y_n) \) in \( X \) convergent to \( y \). Then \( \rho(F(x), F(y_n)) = \rho(F(x), f(y_n)) = d(x, y_n) \). By triangle inequality as above, \( \lim_{n \to \infty} d(x, y_n) = d(x, y) \) and \( \lim_{n \to \infty} \rho(F(x), f(y_n)) = \rho(F(x), F(y)) \). We prove that \( \rho(F(x), F(y)) = d(x, y) \) when \( x, y \in X \setminus D \).

Let us prove that such \( F \) is unique. Let \( F' : X \to Y \) be an isometry so that \( F'|_D = f \). For \( x \in D \), \( F'(x) = f(x) = F(x) \). For \( x \in X \setminus D \), choose a sequence \( (x_n) \) in \( D \) such that \( (x_n) \) is convergent to \( x \). Then

\[
\rho(F(x), F'(x)) \leq \rho(F(x), F(x_n)) + \rho(F(x_n), F'(x_n)) + \rho(F'(x_n), F'(x)) = \rho(F(x), f(x_n)) + d(x_n, x).
\]

By taking \( n \to \infty \), we find \( \rho(F(x), F'(x)) = 0 \). Thus \( F(x) = F'(x) \) when \( x \in X \setminus D \). We conclude that \( F(x) = F'(x) \) for all \( x \in X \). This proves our assertion. \( \square \)

Now let us go back to prove that any two completions of a metric space are isomorphic.

**Proof.** Let \( f = j_2 \circ j_1^{-1} : j_1(X) \to X_2 \). Then \( f \) is an isometry. Since \( X_2 \) is complete, \( f \) can be extended to an isometry \( F : X_1 \to X_2 \). Let us prove that \( F \) is surjective. Let \( y \in X_2 \). Choose a sequence \( (y_n) \) in \( j_2(X) \) so that \( (y_n) \) is convergent to \( y \). Since \( (y_n) \) in \( j_2(X) \), we can choose \( (x_n) \) in \( X \) so that \( y_n = j_2(x_n) \). Take \( z_n = j_1(x_n) \) for any \( n \geq 1 \). Since \( j_1, j_2 \) are isometry, \( (z_n) \) is a Cauchy sequence in \( X_1 \). By completeness of \( X_1 \), we choose \( z = \lim_{n \to \infty} z_n \). Let us show that \( F(z) = y \). By triangle inequality, we can show that \( d_2(F(z), y) = \lim_{n \to \infty} d_2(F(z_n), y) : \)

\[
|d_2(F(z), y) - d_2(F(z_n), y)| \leq d_2(F(z_n), F(z)) = d_1(z_n, z).
\]

Moreover, it follows from the definition that

\[
\lim_{n \to \infty} d_2(F(j_1(x_n)), y) = \lim_{n \to \infty} d_2(j_2(x_n), y) = \lim_{n \to \infty} d_2(y_n, y) = 0.
\]

This implies that \( d_2(F(z), y) = 0 \). Hence \( y = F(z) \). We prove that \( F \) is surjective. \( \square \)

This theorem implies that the completion of a metric space is unique up to isomorphisms.

**Corollary 1.2.** (Universal property of completion of a metric space) Let \( (X, d) \) be a metric space. Given any isometry \( f : X \to Y \) into a complete metric space \( Y \) and any completion \( (\hat{X}, \hat{d}, j) \) of \( (X, d) \) there is a unique isometry \( F : \hat{X} \to Y \) such that \( f = F \circ j \).

**Proof.** Let \( Z = j(X) \). Then \( j : X \to Z \) is an isomorphism. Define \( g : Z \to Y \) by \( f \circ j^{-1} \). Since \( Z \) is dense in \( \hat{X} \) and \( g \) is an isometry, by Lemma 1.1, there is a unique isometry \( F : \hat{X} \to Y \) so that \( F|Z = g \). This is equivalent to say that \( F \circ j = f \). We prove our assertion. \( \square \)