BANACH CONTRACTION MAPPING PRINCIPLE

1. Space of Continuous Functions on Sequentially Compact Spaces

**Theorem 1.1.** Let \( f : M \to N \) be a continuous function. If \( K \) is a sequentially compact subset of \( M \), then \( f(K) \) is a sequentially compact subset of \( N \).

**Proof.** Let \( (y_n) \) be a sequence in \( f(K) \). Then we can choose a sequence \( (x_n) \) in \( K \) such that \( y_n = f(x_n) \) for any \( n \geq 1 \). Since \( K \) is sequentially compact, \( (x_n) \) has a convergent subsequence \( (x_{n_k}) \) in \( K \) whose limit also belongs to \( K \). Denote \( x = \lim_{k \to \infty} x_{n_k} \). Since \( y_{n_k} = f(x_{n_k}) \), by continuity of \( f \),

\[
\lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} f(x_{n_k}) = f(x).
\]

We see that \( (y_{n_k}) \) is convergent to \( f(x) \in f(K) \). Thus we prove that \( (y_n) \) has a subsequence \( (y_{n_k}) \) converging to a point of \( f(K) \). Thus \( f(K) \) is sequentially compact. \( \square \)

**Corollary 1.1.** Let \( K \) be any sequentially compact space and \( f : K \to \mathbb{R} \) be a continuous function. Then \( f \) is a bounded function on \( K \) and \( f(K) \) is a closed subset of \( \mathbb{R} \).

**Proof.** Since \( K \) is sequentially compact, \( f(K) \) is sequentially compact subset of \( \mathbb{R} \). By Bolzano-Weierstrass Theorem, \( f(K) \) is a bounded and closed subset of \( \mathbb{R} \). \( \square \)

**Proposition 1.1.** The space \( C(K, \mathbb{R}) \) of real valued continuous functions on \( K \) is a vector subspace of \( B(K, \mathbb{R}) \) such that \( C(K, \mathbb{R}) \) is a closed subset of \( B(K, \mathbb{R}) \).

**Proof.** The previous corollary implies that \( C(K, \mathbb{R}) \) is a subset of \( B(K, \mathbb{R}) \). We leave it to the reader to check that \( C(K, \mathbb{R}) \) is a vector subspace of \( B(K, \mathbb{R}) \).

Let \( f \) be an adherent point of \( C(K, \mathbb{R}) \). Choose a sequence \( (f_n) \) in \( C(K, \mathbb{R}) \) such that \( (f_n) \) converges to \( f \) in \( B(K, \mathbb{R}) \). To show that \( f \) belongs to \( C(K, \mathbb{R}) \), it is equivalent to show that \( f \) is continuous.

For any \( \epsilon > 0 \), we can choose \( N \in \mathbb{N} \) so that \( \|f_N - f\|_{\infty} < \epsilon/3 \). Let \( x_0 \in M \). Since \( f_N \) is continuous at \( x_0 \), we can choose \( \delta > 0 \) so that \( |f_N(x) - f_N(x_0)| < \epsilon/3 \) whenever \( d(x, x_0) < \delta \). For \( d(x, x_0) < \delta \),

\[
|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|
\]

\[
\leq \|f_N - f\|_{\infty} + \frac{\epsilon}{3} + \|f_N - f\|_{\infty}
\]

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Thus \( f \) is continuous at \( x_0 \) for all \( x_0 \in K \). Thus \( f \in C(K, \mathbb{R}) \). \( \square \)

We also denote the restriction of \( \| \cdot \|_{\infty} \) to \( C(K, \mathbb{R}) \) by \( \| \cdot \|_{\infty} \). Then

**Corollary 1.2.** The norm subspace \( (C(K, \mathbb{R}), \| \cdot \|_{\infty}) \) of \( (B(K, \mathbb{R}), \| \cdot \|_{\infty}) \) is a Banach space over \( \mathbb{R} \).

**Proof.** Since \( (B(K, \mathbb{R}), \| \cdot \|_{\infty}) \) is complete and \( C(K, \mathbb{R}) \) is a closed subset of \( B(K, \mathbb{R}) \). By Homework 7 (2), the normed subspace \( (C(K, \mathbb{R}), \| \cdot \|_{\infty}) \) is also complete. Hence \( (C(K, \mathbb{R}), \| \cdot \|_{\infty}) \) is a Banach space over \( \mathbb{R} \). \( \square \)
2. Banach Contraction mapping principle

Let \((M, d)\) and \((N, \rho)\) be metric spaces. A function \(f : M \to N\) is called a Lipschitz function if there exists \(C > 0\) such that

\[
\rho(f(x), f(y)) \leq Cd(x, y), \quad x, y \in M.
\]

Furthermore, if \(C < 1\), \(f\) is called a contraction mapping.

**Lemma 2.1.** A Lipschitz function \(f : M \to N\) is continuous.

**Proof.** Let \(x_0\) be any point of \(M\). For any \(\epsilon > 0\), we choose \(\delta = \epsilon/C\).

\[
\rho(f(x), f(x_0)) \leq Cd(x, x_0) < C\rho = \epsilon.
\]

Hence \(f\) is continuous at \(x_0\) for any \(x_0 \in M\). Therefore \(f\) is continuous. \(\square\)

**Definition 2.1.** Let \(f : M \to M\) be a function. We say that \(x\) is a fixed point of \(f\) if \(f(x) = x\).

**Theorem 2.1.** Let \(f : M \to M\) be a contraction mapping on a complete metric space. Then \(f\) has a unique fixed point.

**Proof.** Since \(f\) is a contraction, there exists \(0 < C < 1\) so that \(d(f(x), f(x')) \leq Cd(x, x')\). Since any Lipschitz function is continuous, \(f\) is continuous. Choose \(x_0 \in M\) and define a sequence \((x_n)\) by

\[
x_n = f(x_{n-1}), \quad n \geq 1.
\]

By the definition of \((x_n)\) and by the fact that \(f\) is a contraction,

\[
d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq Cd(x_n, x_{n-1}) \text{ for any } n \geq 1.
\]

By homework 6's technique, \((x_n)\) is a Cauchy sequence in \(M\). Since \((M, d)\) is complete, \((x_n)\) is convergent to a point \(x\) in \(M\). By continuity, we find

\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_{n-1}) = f(x).
\]

We prove the existence of fixed points of \(f\).

Suppose \(x, y\) are fixed points of \(f\). If \(x \neq y\), then \(d(x, y) > 0\) and hence

\[
d(x, y) = d(f(x), f(y)) \leq \rho d(x, y) < d(x, y)
\]

which is impossible. Hence the fixed point of \(f\) is unique. \(\square\)

As an application, let us prove a special case for the existence of solution to the following ordinary differential equations. Let \(f : [0, 1] \times \mathbb{R} \to \mathbb{R}\) be a continuous function such that there exists \(0 < C < 1\) with

\[
|f(t, y_1) - f(t, y_2)| \leq C|y_1 - y_2|
\]

for any \(t \in [0, 1]\) and for any \(y_1, y_2 \in \mathbb{R}\). Let us recall the fundamental Theorem of calculus.

**Theorem 2.2.** (The fundamental Theorem of calculus I) Let \(f : [a, b] \to \mathbb{R}\) be a continuous function. For each \(a \leq x \leq b\), define

\[
F(x) = \int_a^x f(t)dt.
\]

Then \(F\) is a \(C^1\) function such that \(F'(x) = f(x)\).
**Theorem 2.3.** (Existence and Uniqueness of Solution of Ordinary Differential Equations) Let \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be as above. The initial value problem
\[
(2.1) \quad x'(t) = f(t, x(t)), \quad x(0) = x_0
\]
has a unique solution in \( C[0, 1] \).

**Proof.** Integrating (2.1), we obtain an integral equation
\[
(2.2) \quad x(t) = x_0 + \int_0^t f(s, x(s))ds.
\]
Define \( T : C[0, 1] \to C[0, 1] \) by
\[
T(x)(t) = x_0 + \int_0^t f(s, x(s))ds.
\]
Then (2.2) has a unique solution if and only if \( T \) has a fixed point.

Let us prove that \( T \) is a contraction mapping. For \( x_1, x_2 \in C[0, 1] \),
\[
T(x_1)(t) - T(x_2)(t) = \int_0^t (f(s, x_1(s)) - f(s, x_2(s)))ds.
\]
For any \( t \in [0, 1] \),
\[
|T(x_1)(t) - T(x_2)(t)| \leq \int_0^t |f(s, x_1(s)) - f(s, x_2(s))|ds
\leq C \int_0^t |x_1(s) - x_2(s)|ds
\leq C \|x_1 - x_2\|_\infty t
\leq C \|x_1 - x_2\|_\infty.
\]
This implies that
\[
\|T(x_1) - T(x_2)\|_\infty \leq C \|x_1 - x_2\|_\infty.
\]
In other words,
\[
d_\infty(T(x_1), T(x_2)) \leq Cd_\infty(x_1, x_2).
\]
Hence \( T \) is a contraction mapping on a complete metric space \((C[0, 1], d_\infty)\). By the contraction mapping principle, \( T \) has a unique fixed point, said \( x \in C[0, 1] \). Since \( x \) is continuous on \([0, 1]\) and \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous, the function \( g : [0, 1] \to \mathbb{R} \) defined by \( g(t) = f(t, x(t)) \) is continuous. By fundamental theorem of calculus, the function \( G : [0, 1] \to \mathbb{R} \) defined by \( G(s) = \int_0^t g(s)ds \) is \( C^1 \) and \( G'(t) = g(t) = f(t, x(t)) \) with \( G(0) = 0 \). Since \( x \) is a fixed point of \( T \),
\[
x(t) = T(x)(t) = x_0 + \int_0^t f(s, x(s))ds = x_0 + G(t),
\]
By fundamental Theorem of calculus, \( x \) is \( C^1 \) and
\[
x'(t) = G'(t) = f(t, x(t)).
\]
Since \( x(0) = x_0 + G(0) = x_0 \). Thus \( x \) solves (2.1). \( \square \)