2. Base For the Zariski Topology of Spectrum of a ring

Let \( X \) be a topological space. A family of open sets \( \mathcal{B} \) is a base for the topology on \( X \) if every open set of \( X \) is a union of elements of \( \mathcal{B} \).

**Proposition 2.1.** The family of basic open sets \( \mathcal{B} = \{ D(f) : f \in A \} \) forms a base for the Zariski topology on Spec \( A \).

**Proof.** Let \( U \) be an open subset of Spec \( A \). Then there exists an ideal \( I \) of \( A \) so that \( U = \text{Spec } A \setminus V(I) \). For each \( x \in U \), we know \( x \notin V(I) \). In other words, \( I \) is not contained in \( x \). Then there exists \( f_x \in I \setminus x \). Hence \( f_x \notin x \) and thus \( x \notin V(f_x) \). It is equivalent to say that \( x \in D(f_x) \).

On the other hand, \( (f_x) \subset I \) and hence \( V(I) \subset V(f_x) \). We see that \( D(f_x) \subset U \). Notice that \( U \subset \bigcup_{x \in U} D(f_x) \subset U \). We conclude that \( U = \bigcup_{x \in U} D(f_x) \). This shows that every open set of \( X \) is a union of elements of \( \mathcal{B} \). By definition, \( \mathcal{B} \) forms a basis for the Zariski topology on Spec \( A \). \( \square \)

**Definition 2.1.** A topological space is quasi compact if and only if every open covering has a finite sub cover.

**Proposition 2.2.** The spectrum Spec \( A \) of a ring is quasi-compact.

**Proof.** Let \( \{ U_\lambda \} \) be an open covering of \( X = \text{Spec } A \). Since \( \{ D(f) : f \in A \} \) forms a base for the Zariski topology of \( X \), we can in fact assume that \( U_\lambda \) is of the form \( D(f_\lambda) \) for some \( f_\lambda \in A \). Then \( \bigcup_\lambda D(f_\lambda) = X \). This implies that \( \bigcap_\lambda V(f_\lambda) = \emptyset \), i.e. \( V(\bigcup_\lambda (f_\lambda)) = V(1) \). Hence \( \bigcup_\lambda (f_\lambda) = (1) \). Hence \( 1 \in \bigcup_\lambda (f_\lambda) \). By definition, \( 1^n \in \bigcup_\lambda (f_\lambda) \) for some \( n \). But \( 1^n = 1 \). We see that \( \bigcup_\lambda (f_\lambda) = (1) \). We can choose \( a_{\lambda_1}, \ldots, a_{\lambda_n} \in A \) and \( f_{\lambda_1}, \ldots, f_{\lambda_n} \) so that \( \sum_{i=1}^n a_{\lambda_i} f_{\lambda_i} = 1 \). This implies that the unit ideal \( (1) = \sum_{i=1}^n (f_{\lambda_i}) \). This implies that \( X = \bigcup_{i=1}^n D(f_{\lambda_i}) \). Hence Spec \( A \) is quasi-compact. \( \square \)

**Corollary 2.1.** Suppose \( A \) is a ring and \( f \in A \). Then \( D(f) \) is also quasi-compact.

**Proof.** We will prove that \( D(f) \) is homeomorphic to Spec \( A_f \), where \( A_f \) is the localization \( S_f^{-1} A \), where \( S_f = \{ f^n : n \geq 0 \} \). Hence \( D(f) \) is the spectrum of a ring. By the previous proposition, \( D(f) \) is quasi-compact. \( \square \)