1. Localization and Spectrum

**Proposition 1.1.** Let $A$ be a ring and $S$ be a multiplicative subset of $A$. The map $A \rightarrow S^{-1}A$ induces a homeomorphism

$$\text{Spec}(S^{-1}R) \rightarrow \{ p \in \text{Spec} A : S \cap p = \emptyset \}$$

where topology of the right hand side is the subspace topology of Spec $A$. The inverse map is given by $p \rightarrow S^{-1}p$.

**Proof.** Let $\varphi : A \rightarrow S^{-1}A$ be the map sending $a$ to $a/1$. Then we have a continuous map $\text{Spec} \varphi : \text{Spec}(S^{-1}A) \rightarrow \text{Spec} A$. For simplicity, denote $\text{Spec} \varphi$ by $h$. Let $p'$ be a prime ideal in $S^{-1}A$. Then $\varphi^{-1}(p')$ is a prime ideal in $A$ so that $\varphi^{-1}(p') \cap S = \emptyset$. If not, there exists $f \in \varphi^{-1}(p') \cap S$. Then $f \in S$ and $f/1 \in p'$. Since $f \in S$, $1/f \in S^{-1}A$. This implies that $1/1 \in p'$, i.e. $S^{-1}A = p'$ which is absurd because $p'$ is a prime ideal. Hence $\text{Im } h \subset \{ p \in \text{Spec } A : S \cap p = \emptyset \}$. Conversely if $p \in \{ p \in \text{Spec } A : S \cap p = \emptyset \}$, then $\varphi(p) = S^{-1}p$ is a prime ideal in $S^{-1}A$. This is because the localization of an integral domain is an integral domain and hence $S^{-1}A/S^{-1}p \cong S^{-1}(A/p)$ is an integral domain. Moreover, $p = \varphi^{-1}(S^{-1}p)$. Therefore $p \in \text{Im } h$. We find $\text{Im } h = \{ p \in \text{Spec } A : S \cap p = \emptyset \}$.

Let $h' : \text{Im } h \rightarrow \text{Spec}(S^{-1}R)$ by $p \rightarrow S^{-1}p$. For $p \in \text{Im } h$, $h \circ h'(p) = h(S^{-1}A) = \varphi^{-1}(S^{-1}p) = p$ and for any $p'$, $h' \circ h(p') = h'(\varphi^{-1}(p')) = S^{-1}(\varphi^{-1}(p')) = p'$ by definition. Hence $h'$ is the inverse of $h$. Now, we only need to show that $h$ is an open mapping.

Let $D(t/s)$ be a standard open subset in $\text{Spec}(S^{-1}A)$. Let us show that $h(D(t/s)) = D(t) \cap \text{Im } h$. Suppose $p \in D(t) \cap \text{Im } h$. Then $p \cap S = \emptyset$ and $t \notin p$. Then $t/s \notin p' = \varphi(p)$. This shows that $p' \in D(t/s)$. In other words, $p = h(p') \subset h(D(t/s))$ Therefore $D(t) \cap \text{Im } h \subset h(D(t/s))$. Suppose that $p \in h(D(t/s))$. Then $p \in \text{Im } h$ and there is $p' \in D(t/s)$ so that $p = \varphi^{-1}(p')$. Since $p \in \text{Im } h$, $p \cap S = \emptyset$. Since $p' \in D(t/s)$, $t/s \notin p'$. Now, we want to show $p \in D(t)$. Suppose not. $t \in p$. Then $t/s \in p'$ which leads to the contradiction that $t/s \notin p'$. Therefore $t \notin p$ and hence $p \in D(t)$. We conclude that $h(D(t/s)) = D(t) \cap \text{Im } h$.

This shows that $h$ is an open mapping. 

**Corollary 1.1.** Let $A$ be a ring and $f \in A$. Then we obtain a homeomorphism

$$\text{Spec } A_f \rightarrow D(f)$$

**Proof.** Let $\varphi : A \rightarrow A_f$ be the localization and $h : \text{Spec } A_f \rightarrow \text{Spec } A$ be its induced map. Then $\text{Im } h = \{ p \in \text{Spec } A : p \cap S_f = \emptyset \}$, where $S_f = \{ f^n : n \geq 0 \}$. By definition, $\text{Im } h = D(f)$. Using Proposition 1.1, $A_f \rightarrow D(f)$ is a homeomorphism.

**Proposition 1.2.** Let $A$ be a ring and $I$ be an ideal. Then the quotient map $A \rightarrow A/I$ induces a homeomorphism

$$\text{Spec}(A/I) \rightarrow V(I) \subset \text{Spec } A$$

**Proof.** The bijection

$$\{ \text{prime ideals of } A/I \} \leftrightarrow \{ \text{prime ideals of } A \text{ containing } I \}$$

\[\text{If } t/s \in p', \text{ then } t/s = t'/s' \text{ for } t'/s' \in S^{-1}p. \text{ Hence there is } s'' \in S \text{ so that } s''(ts' - st') = 0. \text{ Since } t' \in p, s''st' \in p \text{ and } s's't \text{ is thus in } p. \text{ Since } S \cap p = \emptyset, s's'' \notin S. \text{ Since } p \text{ is a prime, we obtain } t \in p \text{ which leads to a contradiction that } t \notin p.\]
implies that the continuous map \( h : \text{Spec}(A/I) \to V(I) \) is a bijection. Here \( h \) is the induced map of the quotient map.

Let us prove that that map is an open mapping. Claim

\[
h(D(s + I)) = D(s) \cap V(I),
\]

where \( s \) is any representative of \( s + I \). Suppose \( p \in D(s) \cap V(I) \). Then \( s \not\in p \) and \( p \) contains \( I \). Then \( s + I \neq I \). Because \( s \not\in p \), \( s + I \not\in p/I \). Hence \( p/I \in D(s + I) \), we see that \( p \in h(D(s + I)) \). We find \( D(s) \cap V(I) \subset h(D(s)) \). Conversely, if \( p \in h(D(s + I)) \), then \( s + I \not\in p/I \). This implies that \( s \not\in p \) and hence \( p \in D(s) \). We see that \( p \in D(s) \cap V(I) \). We obtain \( h(D(s + I)) \subset D(s) \cap V(I) \).

\[ \square \]