1. Grothendieck Group of Abelian categories

Roughly speaking, an abelian category is an additive category such that finite direct sum exists, the kernel and the cokernel of a morphism exist, and the coimage of a morphism is isomorphic to its image (first isomorphism theorem holds). For example, if \( A \) is a noetherian ring, the category of finitely generated (left) \( A \)-modules is an abelian category. A morphism \( f : A \to B \) in an abelian category \( A \) is a monomorphism if \( \ker f = 0 \) and an epimorphism if \( \text{coker} f = 0 \). We say that the a sequence

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]
is exact at \( B \) if \( \ker g = \text{Im} f \). A sequence of morphisms

\[
\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots
\]
is said to be exact if it is exact at all \( A_i \).

The Grothendieck group \( K(\mathfrak{A}) \) of an abelian category \( \mathfrak{A} \) is an abelian group generated by the set \( \{ [A] \} \) of symbols \( [A] \) of objects of \( \mathfrak{A} \) subject to the relations

\[
[A] = [A'] + [A'']
\]
whenever \( 0 \to A' \to A \to A'' \to 0 \) is exact.

**Lemma 1.1.** Let \( \mathfrak{A} \) be an abelian category and \( K(\mathfrak{A}) \) be its Grothendieck group. Then

1. \( [0] = 0 \), and
2. if \( A \cong B \), \( [A] = [B] \), and
3. \( [A \oplus B] = [A] + [B] \).

**Proof.** To prove (1), we consider the exact sequence \( 0 \to 0 \to 0 \to 0 \to 0 \). To prove (2), we consider the exact sequence \( 0 \to 0 \to A \to B \to 0 \). To prove (3), we consider \( 0 \to A \to A \oplus B \to B \to 0 \). \( \square \)

The Grothendieck group \( K(\mathfrak{A}) \) of \( \mathfrak{A} \) can be constructed as follows. Let \( F(\mathfrak{A}) \) be the free abelian group generated by the isomorphism classes of objects of \( \mathfrak{A} \) and \( R(\mathfrak{A}) \) be the subgroup generated by \( [A] - [A'] - [A''] \) whenever \( 0 \to A' \to A \to A'' \to 0 \) is exact. The quotient group \( F(\mathfrak{A})/R(\mathfrak{A}) \) is the Grothendieck group \( K(\mathfrak{A}) \).

**Theorem 1.1.** Let \( A \) and \( B \) be an abelian category \( \mathfrak{A} \). Then \( [A] = [B] \) in \( K(\mathfrak{A}) \) if and only if there exist short exact sequence \( 0 \to C' \to C \to C'' \to 0 \) and \( 0 \to D' \to D \to D'' \to 0 \) such that \( A \oplus C \cong C'' \oplus D \) is isomorphic to \( B \oplus C \oplus D \oplus D'' \).

An exact functor \( F : \mathfrak{A} \to \mathfrak{B} \) on abelian categories is an additive functor such that for any exact sequence \( 0 \to A' \to A \to A'' \to 0 \), the sequence \( 0 \to F(A') \to F(A) \to F(A'') \to 0 \) is also exact.

**Lemma 1.2.** Let \( F : \mathfrak{A} \to \mathfrak{B} \) be an exact functor of abelian categories. Then \( F \) induces a group homomorphism \( F_* : K(\mathfrak{A}) \to K(\mathfrak{B}) \).

**Proof.** Let us define a map \( F : F(\mathfrak{A}) \to F(\mathfrak{B}) \) by \( F([A]) = [F(A)] \) and extend it additively to all elements of \( F(\mathfrak{A}) \). Since \( F \) is additive,

\[
F([A] - [A'] - [A'']) = F([A]) - F([A']) - F([A'']) = [F(A)] - [F(A')] - [F(A'')].
\]
Since \( F \) is exact, the above identity implies that \( F(R(\mathfrak{A})) \subset R(\mathfrak{B}) \). This shows that the map \( F_* : K(\mathfrak{A}) \to K(\mathfrak{B}) \) sending \( [A] + R(\mathfrak{A}) \to [F(A)] + R(\mathfrak{B}) \) is well-defined. This map \( F_* \) can be extend to a group homomorphism. \( \square \)
Let us abuse the use of the notation \([A]\) : the image of \([A]\) in \(K(\mathfrak{A})\) will also be denoted by \([A]\).

**Lemma 1.3.** Let \(F : \mathfrak{A} \to \mathfrak{B}\) be an exact equivalence of categories. Then \(F_{\ast} : K(\mathfrak{A}) \to K(\mathfrak{B})\) is an isomorphism of abelian groups.

**Proof.** Let \(G : \mathfrak{B} \to \mathfrak{A}\) be the inverse of \(F\). Then \(G \circ F = 1_{\mathfrak{A}}\) and \(F \circ G = 1_{\mathfrak{B}}\). One can check that \(G_{\ast} : K(\mathfrak{B}) \to K(\mathfrak{A})\) is the inverse of \(F_{\ast} : K(\mathfrak{A}) \to K(\mathfrak{B})\). \(\Box\)

**Definition 1.1.** Let \(A\) be a noetherian scheme and \(\mathfrak{Coh}(X)\) be the category of coherent sheaves on \(X\). The category \(\mathfrak{Coh}(X)\) is an abelian category. We define the \(K'\)-group of the scheme \(X\) to be the Grothendieck group of \(\mathfrak{Coh}(X)\) :

\[K'(X) = K(\mathfrak{Coh}(X)).\]

Let \(A\) be a commutative noetherian ring and \(X = \text{Spec} A\) be its corresponding noetherian affine scheme. We know that the global section functor 

\[\Gamma : \mathfrak{Coh}(X) \to \mathfrak{Mod}(A)\]

sending \(F \to \Gamma(F)\) gives an exact equivalence of abelian categories whose inverse is given by \(M \mapsto \tilde{M}\). Lemma 1.3 implies that:

**Corollary 1.1.** Let \(A\) be a noetherian ring and \(X = \text{Spec} A\) be its corresponding affine scheme. Then we have a group isomorphism:

\[K'(X) \cong K(\mathfrak{Mod}(A)).\]

Now, let us define the Grothendieck group of a triangulated category.

Let \(\mathcal{C}\) be a triangulated category. The Grothendieck group \(K(\mathcal{C})\) of \(\mathcal{C}\) is the abelian group generated by isomorphism classes of objects of \(\mathcal{C}\) subject to the relation \([A] = [A'] + [A'']\) whenever we have a distinguish triangle \(A' \to A \to A'' \to A'[1]\).

**Theorem 1.2.** Let \(\mathfrak{A}\) be an abelian category and \(D^b(\mathfrak{A})\) be its bounded derived category. Then the natural map 

\[K(\mathfrak{A}) \to K(D^b(\mathfrak{A}))\]

is an isomorphism of abelian groups.

**Proof.** This will be discussed later. \(\Box\)

Let \(X\) be a noetherian scheme and \(D^b(X)\) be the bounded derived category of coherent sheaves on \(X\). Theorem 1.1 implies:

**Corollary 1.2.** We have a natural isomorphism if abelian groups: \(K'(X) = K(D^b(X))\).

Notice that furthermore, if \(X\) is regular, \(K'(X) \cong K^0(X)\). In this case, we obtain 

\[K^0(X) \cong K(D^b(X))\].