1. **Complex Projective Space**

The n-dimensional complex projective space \( \mathbb{P}^n \) over \( \mathbb{C} \) is the space \( \mathbb{C}^{n+1} \setminus \{0\} \) modulo the relation \( x \sim y \) defined below. Two elements \( x \) and \( y \) in \( \mathbb{C}^{n+1} \setminus \{0\} \) are said to be equivalent if there exists \( \lambda \in \mathbb{C} \setminus \{0\} \) so that \( x = \lambda y \). If \( x \) and \( y \) are equivalent, we write \( x \sim y \). The equivalent class of an element \( x = (x_0, \ldots, x_n) \) of \( \mathbb{C}^{n+1} \setminus \{0\} \) is denoted by \( [x] = (x_0 : \cdots : x_n) \). Let \( \pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) be the quotient map. We equip \( \mathbb{P}^n \) with the quotient topology, i.e. \( U \subseteq \mathbb{P}^n \) is open if and only if \( \pi^{-1}(U) \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \). Since \( \pi \) is continuous and \( \mathbb{C}^{n+1} \setminus \{0\} \) is connected (it is path connected and hence connected), \( \mathbb{P}^n = \pi(\mathbb{C}^{n+1} \setminus \{0\}) \) is connected. (Continuous functions send connected sets to connected sets).

The unit sphere

\[
S^{2n+1} = \{(x_0, \ldots, x_n) \in \mathbb{C}^n : \sum_{i=0}^n |x_i|^2 = 1 \}
\]

is well-defined (left to the reader) and is a bijection. Let us prove that

\[
S^{2n+1} \sim \mathbb{P}^n.
\]

Let \( \lambda \) be a nonzero complex number \( 0 \leq \lambda \), let \( U \) be above. The composition \( \varphi \) we can show that \( \varphi \) is open in \( \mathbb{P}^n \). Therefore \( \varphi \) is a homeomorphism.

This map is well-defined (left to the reader) and is a bijection. Let us prove that \( \varphi \) is a homeomorphism.

To show that \( \varphi \) is continuous, we show that \( \varphi^{-1}(V) \) is open in \( U \) for any open set \( V \) of \( \mathbb{C}^n \). Since \( U \) is open in \( \mathbb{P}^n \), to show that \( \varphi^{-1}(V) \) is open in \( U \), we only need to show that \( \varphi^{-1}(V) \) is open in \( \mathbb{P}^n \). To show that \( \varphi^{-1}(V) \) is open in \( \mathbb{P}^n \), we show that \( \pi^{-1}(\varphi^{-1}(V)) \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \). Let \( U \) be as above. The composition \( \varphi \circ \pi : U_1 \to \mathbb{C}^n \) is given by

\[
(\varphi \circ \pi)(x_0, \ldots, x_n) = \left( \frac{x_0}{x_1}, \ldots, \frac{x_n}{x_1} \right).
\]

It is easy for us to see that \( \varphi \circ \pi : U_1 \to \mathbb{C}^n \) is continuous. Hence \( (\varphi \circ \pi)^{-1}(V) \) is open in \( U_1 \) and hence open in \( \mathbb{C}^{n+1} \setminus \{0\} \) (we use the fact that \( U_1 \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \). Since \( \pi^{-1}(\varphi^{-1}(V)) = (\varphi \circ \pi)^{-1}(V) \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \), \( \varphi^{-1}(V) \) is open in \( \mathbb{P}^n \).

Let us prove that \( \varphi \) is an open mapping, i.e. \( \varphi(W) \) is open in \( \mathbb{C}^n \) for any open set \( W \) of \( U \). Since \( W \) is open in \( U \) and \( U \) is an open subset of \( \mathbb{P}^n \), \( W \) is also open in \( \mathbb{P}^n \). Therefore \( W = \pi^{-1}(W) \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \) and hence open in \( \mathbb{C}^{n+1} \). We only need to show that \( \varphi(W) \) is open. In fact, we can show that \( \varphi \circ \pi : U_1 \to \mathbb{C}^n \) is an open mapping. \( \square \)
Since open sets of the form $I_0 \times I_1 \times \cdots \times I_n$ generates the topology on $\mathbb{C}^{n+1}$, where $I_0, \ldots, I_n$ are open subsets of $\mathbb{C}$, open sets of the form $I_0 \times I_1 \times \cdots \times I_n \cap U'$ generates the topology of $U'$.

Hence open sets of the form $I_0 \times I_1 \times \cdots \times I_n$ with $I_i$ open in $\mathbb{C} \setminus \{0\}$ generates the topology of $U'$.

More precisely, $W'$ is a union of open sets of the form $I_0 \times I_1 \times \cdots \times I_n$ where $I_i$ is open in $\mathbb{C} \setminus \{0\}$.

**Lemma 1.1.** Let $f : X \to Y$ be any function. Suppose $\{A_\alpha : \alpha \in \Lambda\}$ is a family of subsets of $X$. Then

$$f \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} f(A_\alpha).$$

Since any union of open sets is open, if we can show that $(\varphi_1 \circ \pi)(I_0 \times I_1 \times \cdots \times I_n)$ is open in $\mathbb{C}^n$ for any open set of the form $I_0 \times I_1 \times \cdots \times I_n$ with $I_i$ open in $\mathbb{C} \setminus \{0\}$, Lemma 1.1 implies that $(\varphi_1 \circ \pi)(W')$ is open for any open subset $W$ of $U_i$ so that $W' = \pi^{-1}(W)$. Let $J_k = \{x_k/x_i : x_i \in I_i, x_k \in I_k\}$ for any $k \neq i$. Then

$$(\varphi_1 \circ \pi)(I_0 \times \cdots \times I_n) = J_1 \times \cdots \times J_{i-1} \times J_{i+1} \times \cdots \times J_n.$$  

If we can show that all $J_k$ are open in $\mathbb{C}$ for any $k \neq i$, by the fact that the product of open sets is open, $J_1 \times \cdots \times J_{i-1} \times J_{i+1} \times \cdots \times J_n$ is open in $\mathbb{C}^n$ and hence $(\varphi_1 \circ \pi)(I_0 \times \cdots \times I_n)$ is open in $\mathbb{C}^n$.

For each $\mu \in I_i$, we see that

$$J_k = \bigcup_{\mu \in I_i} \{x_k/\mu : x_k \in I_k\} = \bigcup_{\mu \in I_i} \mu^{-1}I_k.$$  

Since $I_k$ is open in $\mathbb{C}$, $\mu^{-1}I_k$ is open for any $\mu \in I_i$. Since any union of open sets is open, $J_k$ is open in $\mathbb{C}$. We complete the proof of our assertion. We conclude that $\varphi_i : U_i \to \mathbb{C}^n$ is a homeomorphism for all $0 \leq i \leq n$.

Let $0 \leq j < i \leq n$. Then

$$\varphi_i(U_i \cap U_j) = \{(z_1, \cdots, z_n) : z_j \neq 0\}.$$  

Hence $\varphi_i(U_i \cap U_j)$ is open. The transition functions are given by

$$(\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \mathbb{C}^n$$

$$(\varphi_j \circ \varphi_i^{-1})(z_1, \cdots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_j+1}{z_j}, \ldots, \frac{z_i}{z_j}, \frac{1}{z_j}, \frac{z_i+1}{z_j}, \ldots, \frac{z_n}{z_j}\right).$$  

Hence $\varphi_j \circ \varphi_i^{-1}$ is biholomorphic on $\varphi_i(U_i \cap U_j)$.

Before proving $\mathbb{P}^n$ is Hausdorff, let us recall some basic facts from topology.

**Lemma 1.2.** Let $X$ be a Hausdorff space and $K$ be a compact subset of $X$. Then $K$ is closed.

**Proof.** To show that $Y$ is closed, we show that $X \setminus K$ is open. Let us prove that every point of $X \setminus K$ is an interior point of $X \setminus K$. Let $x \in X \setminus K$. For any $y \in X$, $x \neq y$. Since $X$ is Hausdorff, we can find an open neighborhood $U_y$ of $x$ and an open neighborhood $V_y$ of $y$ such that $U_y \cap V_y = \emptyset$. Then $\{V_y : y \in K\}$ forms an open cover for $K$. Since $K$ is compact, we can find $y_1, \ldots, y_k$ so that $\{V_{y_i} : 1 \leq i \leq k\}$ covers $K$. Take $U = \bigcap_{i=1}^k U_{y_i}$. Then $U$ is an open neighborhood of $x$. Let us show that $U$ is contained in $X \setminus K$.

Let $z \in U$. If $z \in K$, then $z \not\in U_{y_i}$ for some $1 \leq i \leq k$. Then $z \in V_{y_i} \cap U \subseteq V_{y_i} \cap U_{y_i} = \emptyset$ which is impossible. Therefore $z \not\in K$ and hence $z \in X \setminus K$. We see that $U \subseteq X \setminus K$. We find that $x$ is an interior point of $X \setminus K$.

**Corollary 1.1.** Let $X$ be a compact space and $Y$ be a Hausdorff space. Suppose $f : X \to Y$ is a bijective continuous function. Then $f$ is a homeomorphism.

**Proof.** To show that $f$ is a homeomorphism, we only need to prove that $f$ is a closed mapping. Let $A$ be any closed subset of $X$. Since $X$ is compact and $A$ is a closed subset of $X$, $A$ is compact. Since $f$ is continuous and $A$ is compact, $f(A)$ is compact. Since $Y$ is closed and $f(A)$ is compact, $f(A)$ is a closed subset of $Y$.
**Definition 1.1.** A topological space $X$ is normal if given any disjoint closed subsets $E$ and $F$, there exist open neighborhood $U$ of $E$ and $V$ of $F$ such that $U \cap V = \emptyset$.

**Proposition 1.2.** Any compact Hausdorff space is normal.

*Proof.* Let $X$ be a compact Hausdorff space. Let $B$ and $K$ be disjoint closed subset of $X$. Since $X$ is compact, both $B$ and $K$ are also compact. Let $b \in B$. By the proof of Lemma 1.2 and the compactness of $K$, we can choose an open neighborhood $V = \bigcup_{i=1}^{k} V_{y_i}$ of $K$ for some $y_1, \ldots, y_k \in K$ and an open neighborhood $U_b$ of $b$ such that $U_b \cap V = \emptyset$. Since $B$ is compact and $\{U_b \mid b \in B\}$ forms an open cover for $B$, there exist $b_1, \ldots, b_l \in B$ so that $\{U_{b_i} \mid 1 \leq i \leq l\}$ forms an open cover for $B$. Let $U = \bigcup_{i=1}^{l} U_{b_i}$. Then $U$ is an open neighborhood of $B$. Claim $U \cap V = \emptyset$. Let $z \in U \cap V$. Then $z \in V$ and $z \in U_{b_i}$ for some $i$. Hence $z \in V \cap U_{b_i} = \emptyset$ which is not possible. □

**Proposition 1.3.** Let $X$ and $Y$ be compact spaces. Then the product space $X \times Y$ is also compact.

To prove that $\mathbb{F}^n$ is Hausdorff, we show that $\mathbb{F}^n$ is homeomorphic to $S^{2n+1}/S^1$. We will show that $S^{2n+1}/S^1$ is a Hausdorff space.

Define an action of $S^1$ on $S^{2n+1}$ by

$$S^1 \times S^{2n+1} \to S^{2n+1}, \quad (\lambda, x) \mapsto \lambda x.$$ 

Let us show that this action is continuous. Let $\alpha : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ be the function $(\lambda, x) \mapsto \lambda x$. Then $\alpha$ is continuous. The action of $S^1$ on $S^{2n+1}$ is the restriction of $\alpha$ to $S^1 \times S^{2n+1}$; hence it is continuous.

Two points $x$ and $y$ of $S^{2n+1}$ are equivalent if there exists $\lambda \in S^1$ so that $x = \lambda y$. The quotient space of $S^{2n+1}$ modulo this relation is denoted by $S^{2n+1}/S^1$. The equivalent class of $x \in S^{2n+1}$ modulo this relation is denoted by $[x]_{S^1}$. The quotient map $S^{2n+1} \to S^{2n+1}/S^1$ is denoted by $q$.

**Definition 1.2.** A group with a topology is called a topological group if the function

$$G \times G \to G, \quad (a, b) \mapsto ab^{-1}$$

is continuous. Here $G \times G$ is equipped with the product topology.

**Example 1.1.** $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$ with the subspace topology induced from $\mathbb{C}$ is a commutative topological group called a noncompact (one dimensional) complex torus. The subset $S^1$ of $\mathbb{C}^*$ consisting of complex numbers $z$ so that $|z| = 1$ forms a compact subgroup of $\mathbb{C}$.

To show that $S^{2n+1}/S^1$ is Hausdorff, we need the following Lemma.

**Lemma 1.3.** Let $X$ be a compact Hausdorff space and $G$ be a compact topological group. Suppose $G$ acts on $X$ continuously, i.e. the function

$$m : G \times X \to X, \quad (g, x) \mapsto gx$$

is continuous. Then

1. the quotient map $\pi : X \to X/G$ is an open mapping,
2. the quotient map $\pi : X \to X/G$ is a closed mapping,
3. the orbit space $X/G$ is Hausdorff.

*Proof.* At first, let us prove that the quotient map $\pi : X \to X/G$ is an open mapping. To show that $\pi$ is an open mapping, we need to show that $\pi(U)$ is open in $X/G$ for any open set $U$ of $X$. To show that $\pi(U)$ is open in $X/G$, we need to show that $\pi^{-1}(\pi(U))$ is open in $X$. Claim that

$$(1.1) \quad \pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U).$$

Since $g : X \to X$ is a homeomorphism for any $g \in G$, $g(U)$ is open in $X$ for any open subset $U$ of $X$. If the above equation is true, then $\pi^{-1}(\pi(U))$ is open (any union of open subsets of $X$ is open). Let $y \in \pi^{-1}(\pi(U))$, then $\pi(y) \in \pi(U)$ and hence $\pi(y) = \pi(z)$ for some $z \in U$. Therefore $y = gz \in g(U)$ for some $g \in G$. Hence $y \in \bigcup_{g \in G} g(U)$. Hence $\pi^{-1}(\pi(U)) \subseteq \bigcup_{g \in G} g(U)$. Conversely, if $y \in \bigcup_{g \in G} g(U)$,
then \( y \in g(U) \) for some \( g \in G \). Hence \( y = g z \) for some \( z \in U \). Therefore \( \pi(y) = \pi(z) \in \pi(U) \) which implies that \( y \in \pi^{-1}(\pi(U)) \). Hence \( \bigcup_{g \in G} g(U) \subseteq \pi^{-1}(\pi(U)) \). We conclude that (1.1) holds.

Since \( X \) is compact, any closed subset of \( X \) is also compact. Let \( A \) be a closed subset of \( X \). Then \( A \) is compact. Since any product of compact spaces is compact, \( G \times A \) is compact. Since \( G(A) = m(A, A) \) and \( m \) is continuous, \( G(A) = \bigcup_{g \in G} g(A) \) is compact. To show that \( \pi \) is a closed mapping, we show that \( \pi^{-1}(\pi(A)) \) is closed in \( X \). In fact, \( \pi^{-1}(\pi(A)) = G(A) \) is a compact subset of a Hausdorff space \( Y \), it is closed.

The equivalence class of a point \( x \in X \) is the orbit \([x] = Gx = \{ y \in X : y = gx, g \in G \} \). Since \( G \times X \to X \) is continuous and \( G \) is compact, the orbit \( Gx \) is compact. If \([x] \neq [y] \), then \( Gx \) and \( Gy \) are disjoint compact subsets of \( X \). Since \( X \) is Hausdorff, we can find disjoint open sets \( U \) and \( V \) of \( X \) such that \( Gx \subseteq U \) and \( Gy \subseteq V \). Since \( \pi \) is a closed mapping, it is continuous, \( \pi(U) \) is closed in \( \pi(X) \). Hence \([x] \neq [y] \) implies that \( \pi(U) \neq \pi(V) \) which implies that \( \pi(U) \cap \pi(V) = \emptyset \), i.e. \( [x] \neq [y] \) if \( x \neq y \). To show that \( \pi^{-1}(\pi(U)) \) is open, let \( x \in \pi^{-1}(\pi(U)) \). Then \( \pi(x) \in U \). Therefore \( x \in \pi^{-1}(\pi(U)) \).

Since \( X \) is compact and \( \pi \) is a closed mapping, we may choose disjoint open sets \( U \) and \( V \) so that \( Gx \subseteq U \) and \( Gx \subseteq V \). Since \( U \cap W = \emptyset \), \( Gx \cap W = \emptyset \). Hence \([x] \neq [y] \) which implies that \( [x] \neq [y] \). Then \( \pi^{-1}(\pi(U)) \) is open in \( \pi(X) \). Since \( \pi \) is an open mapping, \( \pi^{-1}(\pi(U)) \) is open in \( \pi(X) \). We obtain disjoint open sets \( U', V' \). Since \( Gx \subseteq U \), \( [y] \in \pi(U) = V' \). Let us show that \([x] \notin [y] \). Therefore \( X \) is a Hausdorff space.

This lemma implies that \( \mathbb{S}^{2n+1}/\mathbb{S}^1 \) is a (compact) Hausdorff space.

**Lemma 1.4.** The complex projective space \( \mathbb{P}^n \) is homeomorphic to \( \mathbb{S}^{2n+1}/\mathbb{S}^1 \).

**Proof.** Let \( r : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{S}^{2n+1} \) be the function \( r(z) = \frac{z}{\|z\|} \) for \( z \in \mathbb{C}^{n+1} \setminus \{0\} \). Then \( r \) is continuous. Define \( f : \mathbb{P}^n \to \mathbb{S}^{2n+1}/\mathbb{S}^1 \) by

\[
f([z]) = [r(z)]_{\mathbb{S}^1},
\]

where \( z \) is a representative of \([z]\). Let us check that this map is well-defined. Suppose \( z \) and \( z' \) are equivalent. Then \( z = \mu z' \) for some nonzero complex number \( \mu \). Then

\[
r(z) = \frac{z}{\|z\|} = \frac{\mu z'}{\|\mu\|\|z'\|} = \lambda r(z')
\]

for \( \lambda = \mu/\|\mu\| \). In other words, \( [r(z)]_{\mathbb{S}^1} = [r(z')]_{\mathbb{S}^1} \).

If \( [r(z)]_{\mathbb{S}^1} = [r(z')]_{\mathbb{S}^1} \), then

\[
\frac{z}{\|z\|} = \lambda \frac{z'}{\|z'\|} \implies z = \lambda \|z\| z'/\|z'\|.
\]

Hence \([z] = [z']\). This shows that \( f \) is injective. Let \([x]_{\mathbb{S}^1} \) be a point in \( \mathbb{S}^{2n+1}/\mathbb{S}^1 \). Choose a representative \( x \) of \([x]_{\mathbb{S}^1} \). Then \( x \in \mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1} \setminus \{0\} \) and \( r(x) = x \). Then \( f([x]) = [r(x)]_{\mathbb{S}^1} = [x]_{\mathbb{S}^1} \). We prove that \( f \) is surjective.

By definition, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}^{n+1} \setminus \{0\} & \xrightarrow{r} & \mathbb{S}^{2n+1} \\
\pi \downarrow & & \downarrow q \\
\mathbb{P}^n & \xrightarrow{f} & \mathbb{S}^{2n+1}/\mathbb{S}^1.
\end{array}
\]

To show that \( f \) is continuous, we show that \( f^{-1}(V) \) is open in \( \mathbb{P}^n \) for any open subset \( V \) of \( \mathbb{S}^{2n+1}/\mathbb{S}^1 \). To show that \( f^{-1}(V) \) is open in \( \mathbb{P}^n \), we show that \( \pi^{-1}(f^{-1}(V)) \) is open in \( \mathbb{C}^{n+1} \setminus \{0\} \). By the above commutative diagram,

\[
\pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V) = (q \circ r)^{-1}(V) = r^{-1}(q^{-1}(V)).
\]
Since $V$ is open in $\mathbb{S}^{2n+1}/\mathbb{S}^1$, $q^{-1}(V)$ is open in $\mathbb{S}^{2n+1}$. Since $r$ is continuous and $q^{-1}(V)$ is open in $\mathbb{S}^{2n+1}$, $r^{-1}(q^{-1}(V))$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. This proves that $\pi^{-1}(f^{-1}(V))$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. Hence $f^{-1}(V)$ is open in $\mathbb{P}^n$. We conclude that $f$ is continuous.

Since $f$ is a bijective continuous map and $\mathbb{P}^n$ is compact and $\mathbb{S}^{2n+1}/\mathbb{S}^1$ is Hausdorff, $f$ is a homeomorphism.

**Lemma 1.5.** Let $X$ and $Y$ be two spaces. Suppose $f : X \to Y$ is a homeomorphism. Then $X$ is Hausdorff if and only if $Y$ is Hausdorff.

*Proof.* Let us assume that $X$ is Hausdorff. Suppose $y_1$ and $y_2$ are two points of $Y$ such that $y_1 \neq y_2$. Since $f$ is surjective, there exist $x_1$ and $x_2$ in $X$ so that $f(x_i) = y_i$ for $i = 1, 2$. Since $f$ is a function, $x_1 \neq x_2$. (If $x_1 = x_2$, $y_1 = f(x_1) = f(x_2) = y_2$.) Since $X$ is Hausdorff, there exist open neighborhoods $U_i$ of $x_i$ so that $U_1 \cap U_2 = \emptyset$. Let $V_i = f(U_i)$ for $i = 1, 2$. Then $V_i$ are open neighborhood of $y_i$ for $i = 1, 2$. Claim $V_1 \cap V_2 = \emptyset$. Suppose not. Take $z \in V_1 \cap V_2$. By surjectivity of $f$, we can find $x \in X$ so that $z = f(x)$. Then $x \in f^{-1}(V_1 \cap V_2) \subseteq U_1 \cap U_2 = \emptyset$ which is impossible. □

Since $\mathbb{P}^n$ is homeomorphic to $\mathbb{S}^{2n+1}/\mathbb{S}^1$ and $\mathbb{S}^{2n+1}/\mathbb{S}^1$ is Hausdorff, $\mathbb{P}^n$ is also Hausdorff.