1. A Remark on the space of continuous functions and Square integrable functions

Let (X, Σ, μ) be a measure space and $\mathcal{L}^2(X)$ be the set of all complex-valued Lebesgue measurable functions on X such that

$$\int_X |f(x)|^2 d\mu < \infty.$$

The function $\mathcal{L}^2(X) \to \mathbb{R}$ defined by $f \mapsto (\int_X |f|^2 dx)^{1/2}$ is not a norm on $\mathcal{L}^2(X)$ because there is a nonzero measurable function f such that $\int_X |f|^2 dx = 0$. We therefore consider an equivalence relation on $\mathcal{L}^2(X)$ defined as follows. We say that f is equivalent to g in $\mathcal{L}^2(X)$ if $f = g$ almost everywhere. That is, there exists a measure zero set Z such that $f = g$ on $X \setminus Z$. The quotient space of $\mathcal{L}^2(X)$ modulo the relation is denoted by $L^2(X)$. The quotient space is also a complex vector space: we define

$$[f] + [g] = [f + g], \quad a[f] = [af]$$

where $[f], [g] \in L^2(X)$ and $a \in \mathbb{C}$. We call $L^2(X)$ the space of square integrable functions on X. Let $[f]$ be an equivalent class in $L^2(X)$. We define

$$\|[f]\|^2_{L^2(X)} = \int_X |f(x)|^2 dx$$

for a representative f in $[f]$. This is a well-defined function on $L^2(X)$ and hence we can verify that it gives a norm on $L^2(X)$. Moreever, if we set

$$\langle [f], [g] \rangle = \int_X f(x)\overline{g(x)} d\mu,$$

where f, g are representatives of $[f]$ and $[g]$ respectively. Then $\|[f]\|^2_{L^2(X)} = \langle [f], [f] \rangle$ and $L^2(X)$ becomes a complex Hilbert space.

Let K be a compact subset of $\mathbb{R}^n$. The space of complex-valued continuous functions on K and the space of complex valued Lebesgue square integrable functions are denoted by $C(K)$ and $L^2(K)$ respectively. A continuous function on K is Lebesgue measurable. (They are Borel functions). Moreover, for any $f \in C(K)$, one has

$$\int_K |f(x)|^2 dx \leq \|f\|^2_\infty \int_K 1 dx = |K| \|f\|^2_\infty, \tag{1.1}$$

where $|K|$ is the Lebesgue measure of K. We find that f is also Lebesgue square integrable. Given $f \in C(K)$, we denote $[f]$ its equivalent class in $L^2(K)$. We obtain a map

$$T : C(K) \to L^2(K), \quad f \mapsto [f].$$

T is obviously linear.

Definition 1.1. Let $T : X \to Y$ be a linear operators where X and Y are normed spaces. T is said to be bounded if there exists $M > 0$ such that

$$\|Tx\|_Y \leq M \|x\|_X$$

for all $x \in X$.

By (1.1), $\|T(f)\|_{L^2(X)} \leq M \|f\|_\infty$, for all $f \in C(K)$, where $M = \sqrt{|K|}$ and hence $T : C(K) \to L^2(K)$ is a bounded linear operator. Moreover, if $T(f) = T(g)$ for $f, g \in C(K)$, then $f = g$ almost everywhere on K. Since both f and g are continuous on K and $f = g$ almost everywhere on K, f must be equal to g. If not, assume $f(x_0) \neq g(x_0)$ for some $x_0 \in K$, then there exists an open ball $B(x_0, \delta)$ such that $f(x) \neq g(x)$ on $B(x_0, \delta)$. Since

$B(x_0, \delta)$ has positive measure and $f \neq g$ on $B(x_0, \delta)$, we find that f is not equal to g almost everywhere. This leads to a contradiction to the assumption that $f = g$ almost everywhere. This shows that $\ker T = \{0\}$. Let $V = T(C(K))$ be the image of $C(K)$ under T. We obtain a linear isomorphism $T : C(K) \to V$. We identify $C(K)$ with the linear subspace V of $L^2(K)$ via T. Hence we can think of $C(K)$ as a vector subspace of $L^2(K)$. Similarly, for each $p \geq 1$, we can consider the space $L^p(K)$. We identify $C(K)$ as a vector subspace of $L^p(K)$.