1. Divisors on Riemann surfaces

All the Riemann surfaces in this note are assumed to be connected and compact.

Let \( X \) be a Riemann surface of genus \( g \geq 0 \) and \( K(X) \) be the field of meromorphic functions on \( X \). The free abelian group generated by points of \( X \) is denoted by \( \text{Div}(X) \). Elements of \( \text{Div}(X) \) are called divisors on \( X \). They are of the form

\[
D = n_1 P_1 + \cdots + n_k P_k
\]

for some \( n_1, \cdots, n_k \in \mathbb{Z} \) and for some points \( P_1, \cdots, P_k \in X \) and for some \( k \geq 1 \). Or equivalently, \( D \) can be written as

\[
D = \sum_{P \in X} n_P P,
\]

where \( n : X \to \mathbb{Z} \) is a function so that \( n_P = 0 \) except a finite number of \( P \in X \).

Given a divisor \( D \) of the form (1.1), its degree denoted by \( \text{deg} D \) is defined to be

\[
\text{deg} D = \sum_{P \in X} n_P.
\]

The map \( \text{deg} : \text{Div}(X) \to \mathbb{Z} \) sending a divisor \( D \) to its degree \( \text{deg} D \) is a group homomorphism.

Let \( f \) be a nonconstant meromorphic function on \( X \) and \( P \) be a point of \( X \). In a neighborhood of \( P \), we write \( f = g/h \) with \( f, g \) holomorphic. We set

\[
\text{ord}_P f = \text{ord}_P g - \text{ord}_P h.
\]

We say that \( f \) has a zero of order \( \text{ord}_P f \) if \( \text{ord}_P f > 0 \) and that \( f \) has a pole of order \( \text{ord}_P f \) if \( \text{ord}_P f < 0 \). If \( P \) is neither a zero nor a pole, we set \( \text{ord}_P f = 0 \). We define a divisor

\[
(f) = \sum_{P \in X} (\text{ord}_P f) P.
\]

If \( f = g/h \) with \( g, h \) holomorphic and relatively prime, the divisors of zero \((f)_0\) is defined to be

\[
(f)_0 = \sum_{P} (\text{ord}_P g) P
\]

and the divisor of poles is defined to be

\[
(f)_\infty = \sum_{P \in X} (\text{ord}_P h) P.
\]

Clearly, these divisors are well-defined as long as we require that \( g, h \) are relatively prime, and

\[
(f) = (f)_0 - (f)_\infty.
\]

It is clear that the map \( (\cdot) : K(X)^* \to \text{Div}(X) \) sending \( f \) to its associated divisor \( (f) \) is a group homomorphism. A divisor \( D \) is called principal if there exists \( f \in K(X)^* \) such that \( D = (f) \).

Lemma 1.1. Let \( f \in K(X)^* \). Then \( \text{deg}(f) = 0 \).

Proof. If \( f \) is a constant function, \( (f)_0 = (f)_\infty = 0 \). Hence \( \text{deg}(f) = 0 \) is obvious. Assume that \( f \) is not a constant function. Since \( X \) is compact, there are only finitely many zeros and poles of \( f \) on \( X \). Write \( (f)_0 = n_1 P_1 + \cdots + n_k P_k \) and \( (f)_\infty = m_1 Q_1 + \cdots + m_l Q_l \) where
\[ P_1, \ldots, P_k, Q_1, \ldots, Q_l \in X \text{ and } n_i, m_j \geq 1. \] Since a nonconstant meromorphic function \( f \) on \( X \) is a holomorphic mapping \( f : X \to \mathbb{P}^1 \), the degree of \( f \) is defined to be

\[ \deg f = \sum_{P \in f^{-1}(Q)} e_P. \]

Here \( e_P \) is the ramification index of \( f \) at \( P \) (or the multiplicity). When we choose \( Q = 0 \), we get \( \deg f = n_1 + \cdots + n_k \). When we choose \( Q = \infty \), we obtain \( \deg f = m_1 + \cdots + m_l \).

Then

\[ \deg(f) = (n_1 + \cdots + n_k) - (m_1 + \cdots + m_l) = \deg f - \deg f = 0. \]

A meromorphic one-form \( \omega \) on \( X \) is also called an abelian differential. Given an abelian differential \( \omega \), we can also define its associated divisor

\[ (\omega) = \sum_{P \in X} (\text{ord}_P \omega)P. \]

A divisor of this form is called a canonical divisor. In the next lemma, we are going to see that any two canonical divisors are linearly equivalent.

**Lemma 1.2.** Let \( \omega_1 \) and \( \omega_2 \) be two abelian differentials on \( X \) with \( \omega_1 \neq 0 \). Then there is a unique geometric function \( f \) on \( X \) such that \( \omega_2 = f \omega_1 \). Hence \( \omega_1 \) is linearly equivalent to \( \omega_2 \).

**Proof.** Let \( \phi : U \subset X \to V \subset \mathbb{C} \) be a local chart on \( X \). Write \( \omega_i \circ \phi(z) = g_i(z)dz \) for some meromorphic function \( g_i \) on \( V \). Set \( h = g_2/g_1 \). Then \( f = h \circ \phi \) is a meromorphic function on \( U \). Then \( f \) can be glued to a global meromorphic function on \( X \) such that \( \omega_2 = f \omega_1 \).

This implies that \( \omega_2 = (f)(\omega_1) \). In other words, \( \omega_2 - \omega_1 = (f) \) is a principal divisor. Hence \( \omega_1 \) and \( \omega_2 \) are linearly equivalent. \( \square \)

Notice that by \( \deg(f) = 0 \), we have

\[ \deg(\omega_2) = \deg(\omega_1) = \deg(\omega_1). \]

This formula tells us that every canonical divisor has the same degree. Now, let us compute the degree of a canonical divisor on a compact Riemann surface. In order to do this, we need the following lemma.

**Lemma 1.3.** Let \( f : X \to Y \) be holomorphic mapping between Riemann surfaces and \( \omega \) be a meromorphic one-form on \( Y \). For a point \( p \in X \), we have

\[ \text{ord}_p(f^*\omega) = (1 + \text{ord}_{f(p)} \omega)\text{ord}(f) - 1. \]

**Proof.** Let \( (\psi, U) \) and \( (\varphi, V) \) be local charts around \( p \) and \( f(p) \) with \( f(U) \subset V \), and \( \psi(p) = 0 \), and \( \varphi(f(p)) = 0 \) such that the local representation \( F = \varphi \circ f \circ \psi^{-1} \) of \( f \) is given by the local normal form \( z \mapsto z^n \) where \( m = \text{ord}(f) \). Assume that \( \omega \) is represented by \( \omega_U = (c_k w^k + c_{k+1} w^{k+1} + \cdots) dw \) in the local chart \( (\varphi, V) \) with \( c_k \neq 0 \) and \( k = \text{ord}_{f(p)} \omega \). Hence the local representation of \( F^*\omega \) in local chart \( (\psi, U) \) is represented by

\[
F^*\omega_U = F^*(c_k w^k + c_{k+1} w^{k+1} + \cdots) dF^*w = (c_k z^{nk} + c_{k+1} z^{n(k+1)} + \cdots) dz^n = (c_k z^{nk} + c_{k+1} z^{n(k+1)} + \cdots) n z^{n-1} dz = (nc_k z^{nk+n-1} + nc_{k+1} z^{n(k+1)+n-1} + \cdots) dz.
\]
This implies that $\text{ord}_p(f^*\omega) = nk + n - 1 = (\text{ord}_{f(p)} \omega + 1)e_p(f) - 1$ which proves our assertion.

\[\text{deg}(\eta) = \sum_{p \in X} (\text{ord}_p \eta) = \sum_{p \in X} (\text{ord}_p f^* \omega) = \sum_{p \in X} ((1 + \text{ord}_{f(p)} \omega)e_p(f) - 1) + \sum_{p \in X, f(p) = \infty} ((1 + \text{ord}_{f(p)} \omega)e_p(f) - 1)\]

We know $\text{ord}_q \omega = 0$ if $q \neq \infty$ and $\text{ord}_q \omega = -2$ if $q = \infty$. Hence we have

\[\text{deg}(\eta) = \sum_{p \in X, f(p) \neq \infty} (e_p(f) - 1) - \sum_{p \in X, f(p) = \infty} e_p(f) - 1 + \sum_{p \in X, f(p) = \infty} (e_p(f) + 1)\]

\[= \sum_{p \in X, f(p) \neq \infty} (e_p(f) - 1) + \sum_{p \in X, f(p) = \infty} (e_p(f) - 1) - \sum_{p \in X, f(p) = \infty} (e_p(f) - 1) + \sum_{p \in X, f(p) = \infty} (e_p(f) + 1)\]

\[= \sum_{p \in X} (e_p(f) - 1) - 2 \sum_{p \in f^{-1}(\infty)} e_p(f)\]

\[= \sum_{p \in X} b_p(f) - 2d\]

\[= 2g - 2 + 2d - 2d\]

\[= 2g - 2.\]

Since any two canonical divisors have the same degree, we obtain that $\deg K_X = 2g - 2$. □

Let $f : X \to Y$ be a nonconstant holomorphic mapping between Riemann surfaces. Given a point $q \in Y$, we define the pull back divisor $f^*(q)$ by

\[f^*(q) = \sum_{p \in f^{-1}(q)} e_p(f)p.\]
In general, given a divisor \( D = \sum_{q \in Y} n_q q \), the pull back divisor is defined to be

\[
    f^*(D) = \sum_{q \in Y} n_q f^*(q).
\]

**Lemma 1.4.** Let \( f : X \to Y \) be a nonconstant holomorphic map between Riemann surfaces.

1. \( f^* : \text{Div}(Y) \to \text{Div}(X) \) is a group homomorphism.
2. The pull back of principal divisor is principal. In fact, if \( h \) is a meromorphic function on \( Y \), then \( f^*(h) = (h \circ f) \).
3. If \( X \) and \( Y \) are compact, so that divisors have degrees, we have

\[
    \deg f^* D = \deg f \cdot \deg D.
\]

**Proof.** It follows from the definition that \( f^* \) is a group homomorphism. Let \( h \) be a meromorphic function on \( Y \). Then \( h \circ f \) is a meromorphic function on \( X \). Write \( (h) = \sum_{q \in Y} (\text{ord}_q h) q \). Then

\[
    f^*(h) = \sum_{q \in Y} (\text{ord}_q h) f^* q = \sum_{q \in Y} \sum_{p \in f^{-1}(q)} (\text{ord}_q h) e_p(f)p
\]

\[
= \sum_{p \in X} \text{ord}_p(h \circ f)p = (h \circ f).
\]

Here we use the fact that \( e_p(h \circ f) = e_p(f) \text{ord}_p h \). Thus (b) is proved.

Since \( f^* \) is linear, if \( D = \sum_{q \in Y} n_q q \), then

\[
    f^* D = \sum_{q \in Y} n_q f^* q = \sum_{q \in Y} \sum_{p \in f^{-1}(q)} n_q e_p(f)p.
\]

Hence

\[
    \deg f^* D = \sum_{q \in Y} \sum_{p \in f^{-1}(q)} n_q e_p(f) = \sum_{q \in Y} \left( \sum_{p \in f^{-1}(q)} e_p(f) \right) n_q.
\]

Since \( \sum_{p \in f^{-1}(q)} e_p(f) = \deg f \), we find

\[
    \deg f^* D = \deg f \cdot \sum_{q \in Y} n_q = \deg f \cdot \deg D.
\]

If \( f : X \to Y \) is a nonconstant holomorphic mapping, the ramification divisor of \( f \) is the divisor on \( X \) defined by

\[
    R_f = \sum_{p \in X} b_p(f)p.
\]

The branched divisor of \( f \) is the divisor on \( Y \) defined by

\[
    B_f = \sum_{q \in Y} \left( \sum_{p \in f^{-1}(q)} b_p(f) \right) q.
\]

**Proposition 1.2.** Let \( f : X \to Y \) be nonconstant holomorphic map between Riemann surfaces. Let \( \omega \) be a nonzero meromorphic one forms on \( Y \). Then

\[
    (f^* \omega) = f^*(\omega) + R_f.
\]
Proof. The proof of this equation is based on the fact that
\[ \text{ord}_p f^* \omega = (1 + \text{ord}_{f(p)} \omega) e_p(f) - 1 = (\text{ord}_{f(p)} \omega) e_p(f) + (e_p(f) - 1). \]
We obtain
\[
(f^* \omega) = \sum_{p \in X} (\text{ord}_p f^* \omega) p \\
= \sum_{p \in X} (\text{ord}_{f(p)} \omega) e_p(f) p + \sum_{p \in X} (e_p(f) - 1) p \\
= \sum_{q \in Y} \sum_{p \in f^{-1}(q)} (\text{ord}_q \omega) e_p(f) p + R_f \\
= \sum_{q \in Y} (\text{ord}_q \omega) f^* q + R_f \\
= f^*(\omega) + R_f.
\]

Corollary 1.1. As above, when \( X \) and \( Y \) are compact, then
\[
2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg(R_f),
\]
where \( g(X) \) and \( g(Y) \) are genus of \( X \) and \( Y \) respectively.

Proof. Computing the degree of \( (f^* \omega) \) and using \( \deg(\omega) = 2g(Y) - 2 \), we find
\[
\deg(f^* \omega) = \deg f^*(\omega) + \deg R_f \\
= \deg f \cdot \deg \omega + \deg R_f \\
= (\deg f)(2g(Y) - 2) + \deg R_f.
\]
Since \( \omega \) is a meromorphic one form on \( Y \), \( f^* \omega \) is a meromorphic one form on \( X \). Since the degree of any canonical divisor on \( X \) has degree \( 2g - 2 \), \( \deg(f^* \omega) = 2g - 2 \). We prove our assertion.

Usually we denote \( \deg R_f \) by \( B \). Corollary 1.1 gives us the second proof of the Riemann-Hurwitz formula:
\[
g(X) = (\deg f)(g(Y) - 1) + 1 + \frac{B}{2}.
\]