Floer Homology on Symplectic Manifolds

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Abstract

The Floer homology was invented by A. Floer to solve the famous Arnold conjecture, which gives the lower bound of the fixed points of a Hamiltonian symplectomorphism.

Floer's theory can be regarded as an infinite dimensional version of Morse theory. The aim of this dissertation is to give an exposition on Floer homology on symplectic manifolds. We will investigate the similarities and differences between the classical Morse theory and Floer's theory. We will also explain the relation between the Floer homology and the topology of the underlying manifold.

摘要

著名的Arnold猜想為辛流形上一哈密頓流的不動點的數目給出一個下限。A. Floer創立了Floer同調理論，從而解決了這猜想。

Floer同調理論可視為Morse理論的一個無限維版本。本論文會對辛流形上的Floer同調作一個簡介。我們會討論經典Morse理論與Floer理論的相似和不同之處，並闡釋Floer同調與其所屬流形拓樸的關係。

Acknowledgements

I would like to thank my advisor Prof. Thomas Au Kwok Keung for his encouragement in writing this thesis. I am grateful to all my teachers. They have not only taught me mathematics, but also the way to appreciate it. I also thank all my schoolfellows in the Department of Mathematics and the Institute of Mathematical Sciences for many enlightening discussions and their patience in answering my questions. I have learned so much from them. My deepest gratitude goes to my mother for her love and care throughout my life.

Contents

Introduction

In 1965 V. I. Arnold conjectured in [2] that a symplectic diffeomorphism of a compact symplectic manifold M onto itself possesses at least as many fixed points as a smooth function on M has critical points, whenever this diffeomorphism is homologous to the identity (i.e. a Hamiltonian symplectomorphism). If we require the function to have non-degenerate critical points, i.e. a Morse function, then this number would be the sum of the Betti numbers of M, as is well-known in classical Morse theory. Therefore it is natural to look for a Morse-type theory to solve this version of Arnold's conjecture. If the diffeomorphism is sufficiently near to the identity this was solved by Arnold [1] himself and also A. Weinstein [37]. Without this assumption, there are some scattered results, like Conley-Zehnder [6] for the $2n$-torus case using variational approach and M. Gromov's [15] result of existence of at least one fixed point when $\pi_2(M) = 0$, using pseudo-holomorphic curves. The breakthrough came from Floer's approach of combining these two ideas in a series of papers, notably [11], which proves the Arnold conjecture in the monotone case. His method can be understood as an infinite dimensional version of Morse theory, and is known as Floer homology now. The later development can be regarded as the extension of his work. We will outline his method under some additional assumptions on the second homotopy group of M.

In chapter 1, we introduce some materials of the classical Morse theory, which is very similar to Floer's theory in many ways. Of particular importance are the Morse homology theorem and the Morse inequalities, which are used in the proof of the Arnold conjecture. Readers familiar with classical Morse theory may skip this chapter and proceed to chapter 2 directly, or they may start at section 1.2 instead to skim through Floer's approach of Morse homology.

In chapter 2, we outline the background and the necessary tools needed in the proof of Arnold's conjecture and also the construction of Floer homology groups. This chapter provides an overview of this dissertation and should be read prior to chapter 3 and chapter 4.

In chapter 3, we present the basic knowledge of Fredholm theory which is used to establish the manifold structure of the moduli space of trajectory. We prove the index formula for a Fredholm operator using Maslov index, which is often easier to calculate than the Fredholm index. This gives the local dimension of the Moduli space. The readers can skip the details without affecting their understanding of the whole picture if they are willing to accept some of the technical results.

In chapter 4, we look at the construction of Floer homology groups in a more detailed way. Two important techniques are the gluing argument and the Gromov's compactness theorem, which can often be combined together perfectly to give many of the important results in this chapter. The success of Floer homology comes when we relate it to the Morse homology of a (time-independent) Morse-Smale function, which is obtained by transforming the time-dependent Hamiltonian function to a nice Morse-Smale function by a process known as Floer continuation. Of course the Morse homology groups are much well-understood, and we can extract information about the fixed points from them. In particular the Arnold conjecture follows as an easy corollary after proving that the Floer homology groups are isomorphic to the singular homology groups of M up to a shift of grading.

The remarks are usually some simple observations and notes. They are of secondary importance.

I was invited to the wonderful world of Floer's theory by my advisor about one year ago and I would like to write some notes about it in a way which (I hope) is elementary and down to earth. However due to my lack of knowledge and time I find it impossible to cover everything in detail even if I aim at writing only a very tiny portion of this theory. Nevertheless I still hope these notes are accessible by a graduate or undergraduate student with a basic knowledge of differentiable manifolds and preferably some classical Morse theory.

Chapter 1

Morse Theory

Floer homology can be understood as an infinite dimensional version of the Morse theory. In fact many ideas in Floer homology are analogous to that of the Morse homology. Especially the Morse homology theorem and the Morse inequalities are useful in proving the Arnold conjecture. It is therefore useful to look at the more classical case of Morse theory first. Two good references are [4] and [25], see also [34] for a more analytic approach which is closely related to Floer's theory.

1.1 Introduction

Definition 1.1. *Let M^n be a smooth finite dimensional manifold. Let $f : M \to \mathbb{R}$ be a smooth function, then $p \in M$ is a critical point of f if $df(p) = 0$. It is a* ***Morse function*** *if for any critical point p, the Hessian $Hf(p)$ of f at p is non-degenerate. Equivalently, in* *local coordinates, the determinant of the Hessian matrix at p given by*

$$Hf(p) = (\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$$

is nonzero, where $x_1, \cdots, x_n$ are the local coordinates around p. We will denote the set of critical points of f by $C(f)$.

Definition 1.2. *Let f be a Morse function and p is a critical point of f. Then the* ***Morse index*** *of f at p,*

$$\lambda(p) = \lambda(p, f) := \textit{number of negative eigenvalues of } Hf(p).$$

The following Morse lemma gives a nice local representation of f near a critical point:

Lemma 1.3 (Morse lemma)**.** *Let f be a Morse function and p is a critical point of f. Then there exists an open neighborhood U of p and a chart $h : U \to \mathbb{R}^n$ such that in this coordinates,*

$$f \circ h^{-1}(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$

where $k = \lambda(p)$.

Corollary 1.4. *The critical points of a Morse function are isolated. In particular, if M is compact, there are only finitely many critical points.*

Basically, one wants to study the global topology of a manifold M through a Morse function by extracting the local information from the critical points of f. More precisely, define the sublevel set $M^a := \{x \in M : f(x) \leq a\}$, we would like to see the change in topology (homotopy type) of these sublevel sets when the value of f runs across a critical point. The following two theorems are useful for us to understand these changes.

Theorem 1.5. *If $a < b$ and there is no critical value of f in $[a, b]$, then M^a and M^b are diffeomorphic:*

$$M^a \cong M^b.$$

Moreover, M^a is a deformation retract of M^b, so the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.

Theorem 1.6. *Suppose p is a critical point of f with Morse index λ, $f(p) = c$ and exists $\varepsilon > 0$ such that there is no other critical point in $f^{-1}[c - \varepsilon, c + \varepsilon]$ except p, then the homotopy type of $M^{c+\varepsilon}$ is obtained by attaching a λ-cell $e_\lambda := \{x \in \mathbb{R}^\lambda : ||x|| \leq 1\}$ to $M^{c-\varepsilon}$:*

$$M^{c+\varepsilon} \approx M^{c-\varepsilon} \cup_\phi e_\lambda$$

by an attaching map $\phi : \partial e_\lambda \to M^{c-\varepsilon}$, where ∂e_λ is the boundary of e_λ. (Here $\approx$ means "homotopy equivalence".) In fact, there is a subset $e \subset M^{c+\varepsilon}$ diffeomorphic to e_λ such that $M^{c-\varepsilon} \cup e$ is a deformation retract of $M^{c+\varepsilon}$.

More generally, if there are exactly k critical points $p_1, \cdots p_k \in f^{-1}(c)$, $\lambda(p_i) = \lambda_i$ and $\varepsilon > 0$ is as above, then

$$M^{c+\varepsilon} \approx M^{c-\varepsilon} \cup_{\phi_1} e_{\lambda_1} \cup \cdots \cup_{\phi_k} e_{\lambda_k}.$$

Theorem 1.6 follows more or less from the Morse lemma, at least when there is only one critical value between $c-\varepsilon$ and $c+\varepsilon$. Intuitively, let U be a neighborhood of a critical point p as given by the Morse lemma. The homotopy type of the sublevel set $\{x \in U : f(x) \leq a\}$ in U changes only when the value of f runs across the critical point c and by analyzing the sublevel sets, the change is exactly given by attaching a λ-cell to $M^{c-\varepsilon}$.
Choose a Riemannian metric g on M. The choice of the particular metric is not very important as it turns out that Morse homology is independent of the choice and "most" metric are good (satisfies the Morse-Smale condition). Consider the flow $\psi_t : M \to M$ generated by the negative gradient vector field of f:

$$\begin{cases} \frac{d}{dt}\psi_t(x) = -\nabla f(\psi_t(x)) \\ \psi_0 = \mathrm{id} \end{cases} \tag{1.1}$$

Definition 1.7. *Let p be a critical point of f, then the* ***unstable manifold*** *of p is defined by*

$$W^u(p) := \{x \in M : \lim_{t \to -\infty} \psi_t(x) = p\}$$

and the ***stable manifold*** *of p is defined by*

$$W^s(p) := \{x \in M : \lim_{t \to \infty} \psi_t(x) = p\}.$$

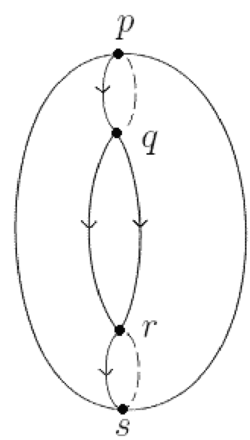
As the names suggest, $W^u(p)$ and $W^s(p)$ are indeed embedded submanifolds of M ([19] corollary 6.3.1).

Definition 1.8. *A Morse function f is said to satisfy the **Morse-Smale condition** if for any critical points p and q of f, $W^u(p)$ and $W^s(q)$ intersect transversally. i.e. for every $x \in W^u(p) \cap W^s(q)$,*

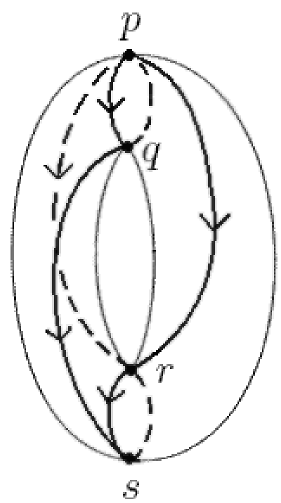
$$T_xW^u(p) + T_xW^s(q) = T_xM.$$

The flow in (1.1) *is then called a **Morse-Smale flow** and (f, g) is called a **Morse-Smale pair**.*

Example 1.9. *Let $f : T^2 \to \mathbb{R}$ be the height function on the torus T^2 (with induced metric g from $\mathbb{R}^3$) as shown in the figure (the arrows denote the directions of the flow):*



This function is not Morse-Smale. The unstable manifold of the saddle point q coincides with the stable manifold of the saddle point r. So at any point of intersection, the two tangent spaces do not span the whole tangent space at that point. However, if we perturb f (or g) a little, this phenomenon will disappear. In fact, $W^u(q)$ and $W^s(r)$ will not even intersect. This can be thought of intuitively as if we tilt the torus a little bit:



In fact the Morse-Smale condition is "generic" (in the sense of Baire).

Theorem 1.10 (Kupka-Smale). *([27], [36]) For a compact smooth Riemannian manifold M, the set of smooth Morse-Smale gradient vector fields is a generic subset of the set X of smooth gradient fields on M.*

Here a subset of X is "generic" means it contains a countable intersection of open dense subset of X (in C^∞ topology). As the Riemannian metric gives a homeomorphism between the space of gradient vector fields and the space of exact one-forms $\{df : f \in C^\infty(M, \mathbb{R})\} \cong C^\infty(M, \mathbb{R})/\mathbb{R}$ on M, it implies that the set of Morse-Smale functions is also a generic subset of $C^\infty(M, \mathbb{R})$.

Definition 1.11. *Let p, q be critical points of a Morse function f. The **space of trajectory** $\mathcal{M}(p,q) = \mathcal{M}(p,q;f,g)$ connecting p and* *q is defined as*

$$\mathcal{M}(p,q) := \{u \in C^\infty(\mathbb{R}, M) : \frac{du}{dt} = -\nabla f(u), \lim_{t\to-\infty} u(t) = p, \lim_{t\to\infty} u(t) = q\}.$$

$\mathcal{M}(p,q)$ naturally embeds into M by

$$\iota : u \mapsto u(0).$$

Under this identification, we have the diffeomorphism ([34] proposition 2.31)

$$\iota : \mathcal{M}(p,q) \cong W^u(p) \cap W^s(q).$$

Theorem 1.12. *Let* (1.1) *be Morse-Smale and p, q are critical points of f. Then $\mathcal{M}(p,q)$ is a smooth manifold and*

$$\dim \mathcal{M}(p,q) = \lambda(p) - \lambda(q).$$

There is a natural action of $\mathbb{R}$ on $\mathcal{M}(p,q)$ by $(\tau, u) \mapsto u(\tau + \cdot)$. Suppose $f(q) < a < f(p)$ and a is not a critical value of f, define $\mathcal{M}^a(p,q) := \iota(\mathcal{M}(p,q)) \cap f^{-1}(a)$. Then by the implicit function theorem and theorem 1.12, $\mathcal{M}^a(p,q)$ is a smooth submanifold of M of dimension $\lambda(p) - \lambda(q) - 1$. It is easy to show the following proposition.

Proposition 1.13. *The map*

$$\begin{aligned} \Psi^a : \mathbb{R} \times \mathcal{M}^a(p,q) &\to \mathcal{M}(p,q) \\ (\tau, x) &\mapsto u(\tau + \cdot) \end{aligned}$$

is a $\mathbb{R}$-equivariant diffeomorphism, where $u(0) = x$ and $\mathbb{R}$ acts by translation on the first factor of $\mathbb{R} \times \mathcal{M}^a(p,q)$.

Definition 1.14. *Define the* ***space of unparametrized trajectory***

$$\hat{\mathcal{M}}(p, q) := \mathcal{M}(p, q)/\mathbb{R}.$$

We will say $u \in \mathcal{M}(p, q)$ is a parametrized trajectory and its image $\hat{u} \in \hat{\mathcal{M}}(p, q)$ an unparametrized trajectory if it is necessary to distinguish these two. By proposition 1.13, we have a diffeomorphism

$$\Psi^a : \hat{\mathcal{M}}(p, q) = \mathcal{M}(p, q)/\mathbb{R} \xrightarrow{\cong} \mathcal{M}^a(p, q).$$

So in particular $\hat{\mathcal{M}}(p, q)$ is also a smooth manifold and

$$\dim \hat{\mathcal{M}}(p, q) = \lambda(p) - \lambda(q) - 1.$$

1.2 Morse Homology

In this section we assume that M is compact and let (1.1) be a Morse-Smale flow on M. We will outline the construction of Morse homology. We do this not only because some of its results are used in Floer's proof of Arnold conjecture, but also because it is very similar to the construction of Floer homology, only simpler. A detailed account can be found in [34] and also [30]. As we will see, many of the arguments here will be carried out again in chapter 4. For simplicity we will work with $\mathbb{Z}_2$ coefficient, so we can ignore the problem of orientation.

When $\lambda(p) - \lambda(q) = 1$, $\hat{\mathcal{M}}(p, q)$ is zero-dimensional, we would like to count the number of points in it. Therefore we have to know that $\hat{\mathcal{M}}(p,q)$ is compact. First we need the notion of a n-dimensional smooth manifold with corners, which is a second countable Hausdorff space such that each point has a neighborhood with a homeomorphism with $\mathbb{R}^{n-k} \times [0,\infty)^k$ for some $0 \leq k \leq n$, and such that the transition maps are smooth. This generalizes the concept of a manifold with boundary and for $n \leq 1$ they are the same. Now recall $C(f)$ is defined to be the set of critical points of f. We have the following compactness result.

Proposition 1.15. *Suppose $p, q \in C(f)$. Then $\hat{\mathcal{M}}(p,q)$ has a natural compactification to a smooth manifold with corners $\overline{\hat{\mathcal{M}}(p,q)}$ by adjoining all the order k broken (unparametrized) trajectories:*

$$\bigcup_{p_0,p_1,\cdots,p_{k+1}\in C(f)} \hat{\mathcal{M}}(p_0,p_1) \times \hat{\mathcal{M}}(p_1,p_2) \times \cdots \times \hat{\mathcal{M}}(p_k,p_{k+1}),$$

*where $p_0 = p$, $p_{k+1} = q$ and all p_i's are distinct. This is called the **compactification by broken trajectories**.*

The proof has two parts. One is a compactness result, which states that any sequence $\hat{u}_n \in \hat{\mathcal{M}}(p,q)$ has a convergence subsequence that converges in an appropriate sense (see theorem 4.3) towards some broken trajectories of order k. The second part is a "gluing argument" which asserts that any order k parametrized broken trajectories can by "glued" together (with a gluing parameter in $[R,\infty)^k$) to form a trajectory in $\mathcal{M}(p,q)$ (see theorem 4.9).

For example, an order one broken trajectories (called simply broken trajectories) is a pair (u, v), where $u \in \mathcal{M}(p, q)$ and $v \in \mathcal{M}(q, r)$, they can be "glued" together at q to get $u\#_\rho v \in \mathcal{M}(p, r)$ for sufficiently large ρ, and $u\#_\rho v$ will converge to (u, v) as $\rho \to \infty$ in some appropriate sense.

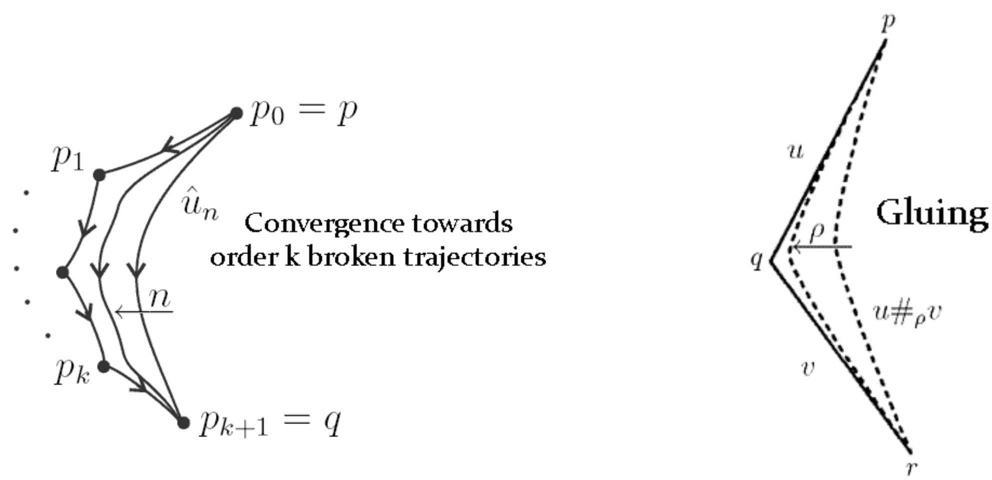


Figure 1.1: Convergence and gluing of simply broken trajectories.

These two arguments can be regarded as the converse of each other. The important consequence of proposition 1.15 is that when $\lambda(p) - \lambda(q) = 1$, then zero-dimensional $\hat{\mathcal{M}}(p, q)$ is compact since there is no broken trajectories to be added for compactification, i.e. it is finite. So at last we are able to make the following definition.

Definition 1.16. *For $p, q \in C(f)$ and $\lambda(p) - \lambda(q) = 1$,*

$$\langle \partial p, q \rangle := \#\hat{\mathcal{M}}(p, q) \pmod 2.$$

Denote $C_k := \mathrm{span}_{\mathbb{Z}_2}\{p \in C(f) : \lambda(p) = k\}$. Then the boundary *operator*

$$\partial_k : C_k \to C_{k-1}$$

is defined by

$$\partial_k p := \sum_{q \in C_{k-1}} \langle \partial p, q \rangle q$$

where $p \in C_k$. *The* ***Morse-Smale-Witten chain complex****, or just the* ***Morse complex****, is defined as* (C_*, ∂_*).

Proposition 1.17. *The boundary operators satisfy*

$$\partial_k \circ \partial_{k+1} = 0.$$

Proof. Let $p \in C_{k+1}$, this statement is equivalent to

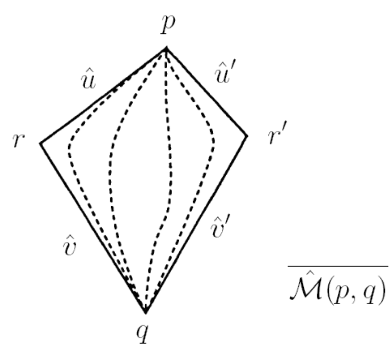
$$\sum_{r \in C_k} \sum_{q \in C_{k-1}} \langle \partial p, r \rangle \langle \partial r, q \rangle q = 0 \pmod 2.$$

So fixing $p \in C_{k+1}$, $q \in C_{k-1}$, we have to prove

$$\sum_{r \in C_k} \langle \partial p, r \rangle \langle \partial r, q \rangle = \# \bigcup_{r \in C_k} \hat{\mathcal{M}}(p, r) \times \hat{\mathcal{M}}(r, q) \tag{1.2}$$

is an even number.

This is proved by the following observation. Each component of the 1-dimensional compact manifold with boundary $\overline{\hat{\mathcal{M}}(p, q)}$ which is not a circle must be a closed bounded interval (having two end-points) by the classification theorem. By proposition 1.15, each end-point of these intervals is of the form of $(\hat{u}, \hat{v}) \in \hat{\mathcal{M}}(p, r) \times \hat{\mathcal{M}}(r, q)$ for some $r \in C_k$.



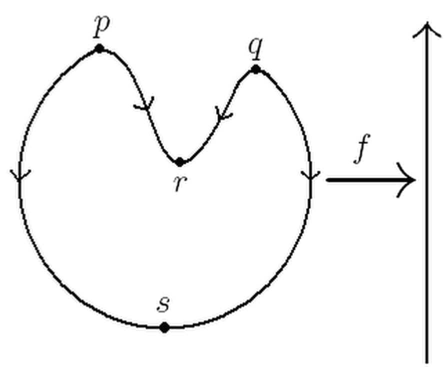
Since there must be an even number of such endpoints in $\overline{\hat{\mathcal{M}}(p,q)}$, it follows that the number in (1.2) is an even number.

$\square$

Definition 1.18. *Define the* ***k-th Morse homology group*** *of* $(M; f, g)$

$$HM_k(M; f, g) := \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

Example 1.19. *Let* $f : \mathbb{S}^1 \to \mathbb{R}$ *be the height function given by the following figure (g can be any metric):*



There are four critical points. The critical points p, q are of index 1 and the critical points r, s are of index 0.

$$\partial p = \partial q = r + s \quad \textit{and} \quad \partial r = \partial s = 0.$$

Therefore

$$HM_1(\mathbb{S}^1; f, g) = \langle p + q \rangle_{\mathbb{Z}_2} \cong \mathbb{Z}_2$$

and

$$HM_0(\mathbb{S}^1; f, g) = \langle r, s \rangle_{\mathbb{Z}_2} / \langle r + s \rangle_{\mathbb{Z}_2} \cong \langle r \rangle_{\mathbb{Z}_2} \cong \mathbb{Z}_2.$$

This agrees with the singular homology $H_*(\mathbb{S}^1)$ of $\mathbb{S}^1$. Although this example is simple, this is not an incident.

Let (f_0, g_0), (f_1, g_1) be two Morse-Smale pairs. It is a remarkable fact that the Morse homology groups $HM_*(M; f_0, g_0)$ and $HM_*(M; f_1, g_1)$ are in fact isomorphic. One approach is to identify each of these Morse homology groups to the singular homology groups of M (see theorem 1.24). However Floer found an elegant alternative approach which establish a more natural isomorphism between $HM_*(M; f_0, g_0)$ and $HM_*(M; f_1, g_1)$, through a process he called continuation, without invoking the singular homology of M. The following explicit construction is from [18].
Let (C^i_*, ∂^i) be the Morse complexes of (f_i, g_i), $i = 0, 1$. The idea is that we can continuously transform (f_0, g_0) to (f_1, g_1) by a smooth homotopy (f_t, g_t), $t \in [0, 1]$, where g_t is a Riemannian metric on M for all t. (Note that the space of all Riemannian metrics on M is contractible.) We then define a vector field $V = V(t, x)$ on $[0, 1] \times M$ by

$$V(t, x) := (1 - t)t(1 + t)\frac{\partial}{\partial t} - \mathrm{grad}_t f_t$$

where $\mathrm{grad}_t f_t$ is the (time-dependent) gradient vector field of $f_t : M \to \mathbb{R}$ with respect to the metric g_t on M. Note that the $(1-t)t(1+t)\frac{\partial}{\partial t}$ is the negative gradient of the function $(t+1)^2(t-1)^2/4$ on $\mathbb{R}$. It is chosen because this function has a critical point of index 1 at $t = 0$ and a critical point of index 0 at $t = 1$ with no critical point in between. Actually this V is the negative gradient vector field of the function $(t+1)^2(t-1)^2/4 + f_t(x)$ on $[0,1] \times M$, where the metric at the point (t,x) is given by the first fundamental form $\begin{pmatrix} 1 & 0 \\ 0 & I(t,x) \end{pmatrix}$, $I(t,x)$ being the first fundamental form of g_t at x . We can define its critical points, stable and unstable manifolds and flow lines just as the case of gradient flow on M before. As before we also require the stable and unstable manifold to intersect each other transversely. If (f_0, g_0) and (f_1, g_1) are Morse-Smale then a generic homotopy between them satisfies this condition. Such homotopy is called **admissible**. However for such homotopy, it may (and often must) happen that for some time $t \neq 0, 1$, the pair (f_t, g_t) is not Morse-Smale on M.

Observe that for $t = 0, 1$, the flow on $[0,1] \times M$ is the same as the flow on M of (f_0, g_0) and (f_1, g_1) respectively. Note also that there are only two kinds of critical points of V, one is of the form $(0,p)$ where $p \in C^0_*$ and the other is of the form $(1,q)$ where $q \in C^1_*$. Also the index of $(0,p)$ is $1 + \lambda(p, f_0)$ and the index of $(1,q)$ is $\lambda(q, f_1)$. Thus $\hat{\mathcal{M}}((0,p),(1,q))$ is zero-dimensional if $\lambda(p, f_0) = \lambda(q, f_1) = k$

and is compact. We then define $\phi : C_k^0 \to C_k^1$ by

$$\phi(p) := \sum_{q \in C_k^1} \langle \phi p, q \rangle q$$

where $\langle \phi p, q \rangle := \#\hat{\mathcal{M}}((0,p),(1,q)) \pmod 2$, i.e. the number of unparametrized trajectories in $[0,1] \times M$ connecting $(0,p)$ and $(1,q)$ modulo two.

Proposition 1.20. *ϕ is a chain map. i.e.*

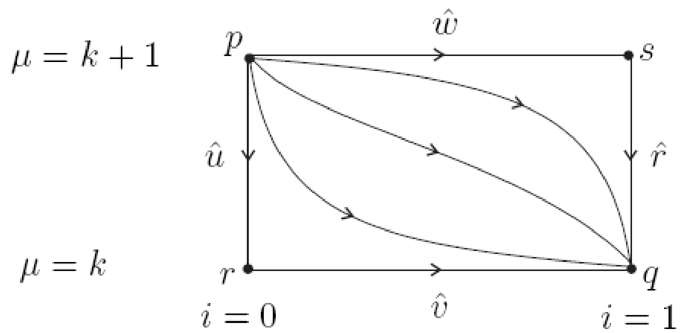
$$\partial^1 \circ \phi = \phi \circ \partial^0.$$

Proof. The proof is similar to that of 1.17. Denote $\hat{\mathcal{M}}_i(p,q) := \hat{\mathcal{M}}(p,q; f_i, g_i)$ for $i = 0, 1$. $\hat{\mathcal{M}}_i(p,q)$ can be naturally identified with $\hat{\mathcal{M}}((i,p),(i,q))$. Let $p \in C_{k+1}^0$ and $q \in C_k^1$, we have to show

$$\sum_{r \in C_k^0} \langle \partial^0 p, r \rangle \langle \phi r, q \rangle = \sum_{s \in C_{k+1}^1} \langle \phi p, s \rangle \langle \partial^1 s, q \rangle. \pmod 2 \tag{1.3}$$

Equivalently, there is an even number of pairs of unparametrized simply broken trajectories between p and q. By proposition 1.15, there are two kinds of endpoints of the one-dimensional compact manifold $\overline{\hat{\mathcal{M}}((0,p),(1,q))}$. One kind is in the form of $(\hat{u}, \hat{v}) \in \hat{\mathcal{M}}_0(p,r) \times \hat{\mathcal{M}}((0,r),(1,q))$, where $r \in C_k^0$. This corresponds to the term on the left of equation (1.3), the other kind of endpoint is of the type $(\hat{w}, \hat{r}) \in \hat{\mathcal{M}}((0,p),(1,s)) \times \hat{\mathcal{M}}_1(s,q)$ with $s \in C_{k+1}^1$, which corresponds to the right hand side of (1.3). Since there must be an even number of endpoints, the result follows.

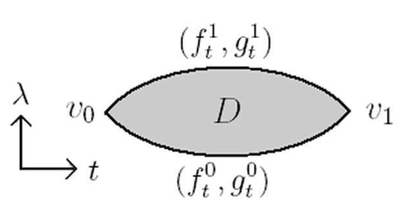
$\square$



Proposition 1.21. *Suppose* (f_t^0, g_t^0) *and* (f_t^1, g_t^1) *are two smooth homotopies between* (f_0, g_0) *and* (f_1, g_1) *as above, and* ϕ^0, ϕ^1 *respectively denotes their induced chain maps. Then* ϕ^0 *and* ϕ^1 *are chain homotopic, i.e. there exists* $\Psi = \Psi_k : C_k^0 \to C_{k+1}^1$ *such that*

$$\phi^0 - \phi^1 = \partial^1 \circ \Psi + \Psi \circ \partial^0.$$

Sketch of proof. Again the proof is similar to that of proposition 1.17. First find a smooth λ-homotopy $(f_t^\lambda, g_t^\lambda)$ between (f_t^0, g_t^0) and (f_t^1, g_t^1), $\lambda \in [0,1]$, such that $(f_i^\lambda, g_i^\lambda) = (f_i, g_i)$ for all λ and $i = 0, 1$. Then $(f_t^\lambda, g_t^\lambda)$ can be regarded as a family $\{(f_d, g_d) : d \in D\}$ parametrized by the 2-gon $D := [0,1] \times [0,1]/\{(0,\lambda_1) \sim (0,\lambda_2)$ and $(1,\lambda_1) \sim (1,\lambda_2)\}$.



Again we find a function $h : D \to \mathbb{R}$ with an index two critical point at the vertex $v_0 := \{0\} \times [0,1]$ and an index 0 critical point at the vertex $v_1 := \{1\} \times [0,1]$ with no other critical point and the negative gradient $-\nabla h(d)$ of h at the two edges are $(1-t)t(1+t)\frac{\partial}{\partial t}$. Define the vector field $V = V(d, x)$ on $D \times M$ by

$$V = -\nabla h - \mathrm{grad}_d f_d$$

where $\mathrm{grad}_d f_d$ is the gradient vector field of f_d with respect to the metric g_d. For a generic choice of (f_d, g_d), V is Morse-Smale. The only critical points of V on $D \times M$ are of the type (v_0, p) with index $\lambda(p, f_0) + 2$ where $p \in C_k^0$ and (v_1, q) with index $\lambda(q, f_1)$ where $q \in C_k^1$. Define $\Psi : C_k^0 \to C_{k+1}^1$ by

$$\Psi p := \sum_{q \in C_{k+1}} \#\hat{\mathcal{M}}((v_0, p), (v_1, q))q.$$

By analyzing the endpoints of the one-dimensional manifold $\hat{\mathcal{M}}((v_0, p)(v_1, r))$ for $p \in C_k^0$, $r \in C_k^1$, we get the result. $\square$

Proposition 1.21 shows that there exists a homomorphism of the Morse homology groups

$$\phi_* : HM_*(M; f_0, g_0) \to HM_*(M; f_1, g_1).$$

If γ_1 is a homotopy from (f_0, g_0) to (f_1, g_1) as above, we denote the induced chain map by ϕ_{γ_1}. Let γ_2 be a homotopy from (f_1, g_1) to (f_2, g_2). Then we can concatenate the two paths, which by reparametrizing and perturbing it if necessary, can be assumed to be a smooth admissible homotopy from (f_0, g_0) to (f_2, g_2), call it $\gamma_2 * \gamma_1$.

Proposition 1.22. *$\phi_{\gamma_2 * \gamma_1}$ and $\phi_{\gamma_2} \circ \phi_{\gamma_1}$ are chain homotopic.*

This is again proved by the compactness-gluing argument and is omitted here. For a constant homotopy from (f, g) to itself, the induced homomorphism of homology is obviously the identity. For two pairs (f_0, g_0) and (f_1, g_1) and any homotopy $\gamma = (f_t, g_t)$ between them, since the inverse homotopy compose with it is homotopic is identity, therefore by the previous proposition each such ϕ_{γ_*} is an isomorphism. Therefore

Theorem 1.23. *For two Morse-Smale pairs (f_0, g_0) and (f_1, g_1) on M, the corresponding Morse homology groups are isomorphic*

$$HM_*(M; f_0, g_0) \cong HM_*(M; f_1, g_1).$$

So we can speak of "the" Morse homology of M without actually specifying a particular Morse-Smale pair. Furthermore, it is actually the same as singular homology of M.

Theorem 1.24 (Morse homology theorem)**.** *The Morse homology is isomorphic to the singular homology of M*

$$HM_k(M; f, g) \cong H_k(M; \mathbb{Z}_2).$$

There are many proofs, see for example [12], [30], [38]. One idea is to relate the singular homology of M with that of a CW complex. We can build a CW complex K whose k-cells corresponds to the critical points of f with Morse index k as follows. Since

M is compact there exists $c_0 < \cdots < c_l$ such that c_i's are all the critical values of f. Suppose a is not a critical value with $c_{k-1} < a < c_k$ and that M^a has the homotopy type of a CW complex, that is M^a is homotopy equivalent to a CW complex K. By theorem 1.5 and 1.6, $M^{c_k+\varepsilon}$ is homotopy equivalent to $M^a \cup_{\phi_1} e_{\lambda_1} \cup \cdots \cup_{\phi_j} e_{\lambda_j}$ where $\lambda_1, \cdots, \lambda_j$ are exactly the indices of the j critical points corresponding to c_k. Then M^{c_k} is homotopy equivalent to $K \cup_{\psi_1} e_{\lambda_1} \cup \cdots \cup_{\psi_j} e_{\lambda_j}$ for some gluing maps $\psi_i : \partial e_{\lambda_i} \to K$. (See [25]).
M^a is empty if $a < c_0$ and by induction M^a has the homotopy type of a CW complex. Let $a_0, \cdots, a_l$ are such that $c_0 < a_0 < c_1 < \cdots < c_l < a_l$, then there is a sequence of homotopy equivalences

$$\begin{array}{ccccccc} M^{a_0} & \subset & M^{a_1} & \subset \cdots \subset & M^l & = M \\ \downarrow & & \downarrow & & \downarrow & \\ K_0 & \subset & K_1 & \subset \cdots \subset & K_l & = K \end{array}$$

each extending the previous one. So M is homotopy equivalent to the CW complex K. Then the singular homology of M is isomorphic to the cellular homology of K. Both the CW complex and the Morse complex are generated by the critical points of f graded by the indices, furthermore it can be proved that the boundary operator of the CW complex and that of the Morse complex are the same (after identification). Intuitively, it is because the "attaching degree" of a k-cell relative to a $(k-1)$-cell is equal to the number of components of the intersection between the unstable manifold of the corresponding index-k critical point and the stable manifold of the corresponding critical point with index $k-1$ (mod 2). The following is a corollary of the Morse homology theorem.

Theorem 1.25 (Weak Morse inequalities)**.** *Let M^n be a compact manifold. Let c_k denotes the number of critical points of index k of f and $b_k := \dim H_k(M,\mathbb{Q})$ denotes the k-th Betti number. Then*

$$b_k \leq c_k \quad \textit{and}$$
$$\chi(M) = \sum_{k=0}^{n}(-1)^k b_k = \sum_{k=0}^{n}(-1)^k c_k.$$

In particular the number of critical points of f is bounded below by the sum of the Betti numbers:

$$\#C(f) = \sum_{k=0}^{n} c_k \geq \sum_{k=0}^{n} b_k.$$

We also have the stronger inequalities.

Theorem 1.26 (Morse inequalities)**.** *For a compact manifold,*

$$b_k - b_{k-1} + \cdots + (-1)^k b_0 \leq c_k - c_{k-1} + \cdots + (-1)^k c_0 \quad \textit{for all } k = 0, \cdots, n$$

For the proof, see for example [25].

Chapter 2

Symplectic Fixed Points and Arnold Conjecture

2.1 Introduction

Let (M^{2n}, ω) be a connected $2n$-dimensional compact symplectic manifold without boundary, i.e. ω is a closed non-degenerate 2-form on M. Then ω determines an isomorphism $I_\omega : T^*M \xrightarrow{\cong} TM$, namely for $\alpha \in T_p^*M$, $\alpha \mapsto v$, where $v \in T_pM$ is the unique vector satisfying $\alpha = \omega_p(v, \cdot)$. Let $H = H(t,x) : \mathbb{R} \times M \to \mathbb{R}$ be a smooth function on M, called a **Hamiltonian function**, such that it is periodic in time (periodic means 1-periodic unless otherwise stated):

$$H(t,x) = H(t+1,x).$$

Then $H_t = H(t, \cdot)$ can be regarded as a time dependent periodic family of function on M. The image of the one form $-dH_t$ under I_ω, denoted by X_t, is called the **Hamiltonian vector field** generated by H_t. That is,

$$-dH_t = \omega(X_t, \cdot).$$

Consider the Hamiltonian system of ordinary differential equations

$$\dot{x}(t) = X_t(x(t)). \tag{2.1}$$

The solutions for (2.1) generates a flow $\psi_t : M \to M$:

$$\begin{cases} \frac{d}{dt}\psi_t = X_t(\psi_t) \\ \psi_0 = \mathrm{id} \end{cases}$$

Let $\psi = \psi_1$ be the time 1 map. Clearly, the fixed points of ψ corresponds to the periodic solutions to (2.1).

Definition 2.1. *Define*

$$P(H) := \{\textit{periodic solutions of } (2.1)\}.$$

As the solutions are periodic, we can also define it as $P(H) := \{x : \mathbb{R}/\mathbb{Z} \to M \mid x \textit{ solves } (2.1)\}$.

By identifying x with $x(0)$, sometimes we will use x to denote either a periodic solution of (2.1) or a fixed point of ψ. We will also identify $\mathbb{R}/\mathbb{Z}$ with $\mathbb{S}^1$ throughout.

Remark 2.2. *For all* t, ψ_t *is a symplectomorphism, i.e.* $\psi_t^*\omega = \omega$, *as* $\psi_0^*\omega = \omega$ *and by Cartan's formula,*

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^*(L_{X_t}\omega) = \psi_t^*(d\iota_{X_t}\omega + \iota_{X_t}d\omega) = \psi_t^*(d\iota_{X_t}\omega) = \psi_t^*(-ddH_t) = 0,$$

where L_{X_t} denotes the Lie derivative along X_t. A symplectomorphism generated by a Hamiltonian vector field is called a ***Hamiltonian*** *or* ***exact symplectomorphism****.*

Definition 2.3. *A fixed point x is called* ***non-degenerate*** *if*

$$\det(I - d\psi(x(0))) \neq 0.$$

i.e. 1 is not an eigenvalue of $d\psi(x(0))$. H is said to be ***regular*** *if all its corresponding fixed points are non-degenerate.*

Arnold conjectured that the number of non-degenerate periodic solutions to this equation is at least the sum of the Betti numbers of M.

Conjecture 2.4 (Arnold conjecture)**.** *Suppose all the periodic solutions of* (2.1) *are non-degenerate. Then*

$$\#P(H) \geq \sum_{i=0}^{2n} b_i$$

where $b_i = \dim H_i(M, \mathbb{Q})$ is the i-th Betti number of M.

Remark 2.5. *1. A fixed point x can be identified with the point (x,x)at the intersection of the graph $\Gamma := \{(x, \psi(x)) : x \in M\}$ of ψ with the diagonal $\Delta := \{(x,x) : x \in M\}$ in $M \times M$. Then x is non-degenerate if and only if Γ intersects with Δ*

transversely at (x,x)*:*

$$\begin{aligned} T_{(x,x)}(M\times M) &= T_{(x,x)}\Gamma + T_{(x,x)}\Delta \\ &= T_{(x,x)}\Gamma \oplus T_{(x,x)}\Delta \quad (\textit{i.e. } T_{(x,x)}\Gamma \cap T_{(x,x)}\Delta = 0) \\ &= \{(v, d\psi_x(v)) : v \in T_xM\} \oplus \{(v,v) : v \in T_xM\} \end{aligned}$$

$\Leftrightarrow d\psi_x(v) \neq v$ *for non-zero* $v \in T_xM$*. i.e.* x *is non-degenerate.*

2. *Non-degenerate* $x \in P(H)$ *are isolated: by choosing a suitable local coordinates, we can regard* $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *such that* x *has local coordinates* 0 *and so* $\psi(0) = 0$*. Then* $d(\psi - id)(0)$ *has non-zero determinant and thus* $\psi - id$ *is a local diffeomorphism around* 0 *by inverse mapping theorem. So locally* $\psi(x) \neq x$ *except* $x = 0$*. Therefore for compact* M*,* $P(H)$ *consists of finite number of points.*

Remark 2.6. 1. *Since* ψ *is isotopic to the identity map, by the Lefschetz fixed point theorem, the number of fixed points of* ψ *is greater than or equal to* $|\sum_{i=0}^{2n}(-1)^i b_i|$*. So Arnold conjecture gives a stronger estimate in this case.*

2. *The comparison of Lefschetz fixed point theorem with Arnold conjecture is analogous to that of Poincare-Hopf theorem, which states that for a smooth vector field* V *on* M *with non-degenerate zeroes,*

$$\#\{x \in M : V(x) = 0\} \geq |\sum_{i=0}^{2n}(-1)^i b_i|,$$

with Morse theory, which states that for a gradient vector field ∇f induced by f (by giving M a Riemannian metric),

$$\#\{x \in M : \nabla f(x) = 0\} = \#\{x \in M : df(x) = 0\} \geq \sum_{i=0}^{2n} b_i.$$

The last inequality comes from theorem 1.25, the weak Morse inequality. Actually the original statement of Arnold is that every Hamiltonian symplectomorphism on M has at least as many fixed points as a function on M has critical points (see [1], [2]), this is clear in particular when H is a time independent Morse function:

3. *For the special case where $H_t \equiv H$, i.e. H is independent of t. Then*

$$\begin{aligned} x \textit{ is a critical point of } H &\Leftrightarrow dH(x) = 0 \\ &\Leftrightarrow X_H(x) = 0 \\ &\Leftrightarrow x(t) \equiv x \in P(H). \end{aligned}$$

In particular if H is a Morse function, then $\#P(H) \geq \sum_{i=0}^{2n} b_i$.

The Arnold conjecture of the above form has now been proved in full generality. Floer ([7], [8], [9], [12]) invented the Floer homology for the monotone case, which is the analogue of Morse homology on finite dimensional smooth manifolds, by studying the "gradient flow" of a certain action functional on the loop space of M. There is also another version of Arnold conjecture for degenerate fixed points (see [16], [10]).

2.2 The Variational Approach

According to classical Morse theory, the existence problem of closed geodesics is restated by the variational approach as the existence of the critical points (which are by definition loops in M) of the energy functional E

$$E(x) := \int_{\mathbb{S}^1} |\dot{x}|^2 dt$$

for $x : \mathbb{S}^1 \to M$ in some appropriate loop space of M. One then is naturally led to apply the same method to study the critical points of the action functional associated to a Hamiltonian system:

$$A(x) = -\int_{\mathbb{D}} u^*\omega + \int_{\mathbb{S}^1} H_t(x(t))dt \tag{2.2}$$

where $u|_{\partial\mathbb{D}} = x : \mathbb{S}^1 \to M$.

A natural inner product structure is introduced on the appropriate loop space so as to define the gradient of A. Then the zeroes of the gradient of A, i.e. its critical points can be identified to the solutions of the Hamiltonian equation (2.1).

However the classical Morse theory approach fails in this infinite dimensional setting due to several reasons.

Unlike the energy functional, the action functional is both unbounded above and below, so there is no absolute minimum or maximum which we can start a Morse complex for cellular decomposition. Moreover, the "Morse index" would not be finite as in the classical cases, as the subspaces on which the Hessian is positive or negative definite are both infinite dimensional. Finally the gradient of the action functional grad A defined above does not give a well-defined flow on the loop space we considered.

However there is still hope. Floer realized that the essential conditions for Morse theory is still satisfied if we reduce it to the relative gradient flow, that is a flow between two fixed critical points x and y of A. We also use a relative Morse index which, roughly speaking, measures the codimension of the "unstable manifold" of y with respect to the "unstable manifold" of x. Floer found the right analytical setup to analyze the space $\mathcal{M}(x, y)$. He then used the structures of these spaces to extract an invariant which is now called Floer homology.

2.3 Action Functional and Moduli Space

Definition 2.7. *The* ***contractible loop space*** $\mathcal{L}$ *of* M *is defined to be all the contractible loops in* M*. i.e.* $\mathcal{L} := \{x \in C^\infty(\mathbb{R}/\mathbb{Z}, M) \mid x \text{ is contractible}\}$.

Denote $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ to be the closed unit disk. So for $x \in \mathcal{L}$, there exists an extension $u : \mathbb{D} \to M$ such that $u(e^{i2\pi t}) = x(t)$.

We will assume throughout that ω vanish over the second homotopy group $\pi_2(M)$ of M. i.e.

$$\int_{\mathbb{S}^2} v^*\omega = 0 \tag{2.3}$$

for any smooth $v : \mathbb{S}^2 \to M$, noting that this integral depend only on the homotopy class of v. This assumption is needed for the well-definedness of the action functional on M and is also crucial for the compactness of the so called moduli space. A symplecic manifold with this condition is called **aspherical** and we will denote this condition as $\omega(\pi_2(M)) = 0$.

Definition 2.8. *The* ***action functional*** $A = A_H : \mathcal{L} \to \mathbb{R}$ *is defined by*

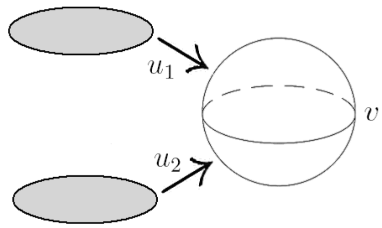
$$A_H(x) := -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))dt,$$

where $u : \mathbb{D} \to M$ *is an extension of* x *to the unit disk.*

Remark 2.9. *A is well defined by the following reason. Suppose u_1, u_2 both extends x, then we can "glue" the two maps along their boundary to get a map from $\mathbb{S}^2$ to M. More precisely, let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere and let $\pi : (x, y, z) \mapsto (x, y)$ by the projection onto the $x - y$ plane. Define $v : \mathbb{S}^2 \to M$ by*

$$v(p) = \begin{cases} u_1(\pi(p)) & \textit{if } p \textit{ is on the upper hemisphere,} \\ u_2(\pi(p)) & \textit{if } p \textit{ is on the lower hemisphere.} \end{cases}$$

Then v is a well defined continuous map and $\int_{\mathbb{S}^2} u_1^*\omega - u_2^*\omega = \int_{\mathbb{S}^2} v^*\omega = 0$. Therefore A is a well defined function.



It turns out that Arnold conjecture is easier to prove in the so called monotone case. Here we will make an even stronger assumption. Let J be an almost complex structure which is compatible with ω, i.e. $J \in C^\infty(\mathrm{End(TM)})$, $J^2 = -I$ and

$$g(\xi, \eta) = \langle \xi, \eta \rangle := \omega(\xi, J\eta), \quad \xi, \eta \in T_xM, \tag{2.4}$$

defines a Riemannian metric on M. Then by the symmetry of g, both ω and g are J-invariant, i.e. $\langle J\xi, J\eta \rangle = \langle \xi, \eta \rangle$ and $\omega(J\xi, J\eta) = \omega(\xi, \eta)$. Such J exists in abundance and in fact the space $\mathcal{J}$ of all compatible almost complex structures of M is contractible (see for example [23]). Then (TM, J) is a complex vector bundle over M with first Chern class $c_1 = c_1(TM, J) \in H^2(M, \mathbb{Z})$. c_1 is independent of the choice of J as we can join two such complex structures J_1, J_2 by a path and thus induce an isomorphism between (TM, J_1) and (TM, J_2) as complex vector bundle.

We will assume throughout, as in (2.3), that c_1 vanishes on $\pi_2(M)$:

$$\int_{\mathbb{S}^2} v^* c_1 = 0 \tag{2.5}$$

for any $v : \mathbb{S}^2 \to M$. This assumption, denoted by $c_1(\pi_2(M)) = 0$, is needed to give a well-defined Maslov type index for the critical point of A and thus a grading of the Floer homology groups. Let us state again our assumptions.

Assumption 2.10. *We will assume throughout that both ω and c_1 vanishes on $\pi_2(M)$:*

$$\int_{\mathbb{S}^2} v^*\omega = 0 \quad \text{and} \quad \int_{\mathbb{S}^2} v^* c_1 = 0$$

for any $v \in C^\infty(\mathbb{S}^2, M)$.

Remark 2.11. 1. *Floer [11] actually proved the Arnold conjecture in the more general case where M is monotone. This means*

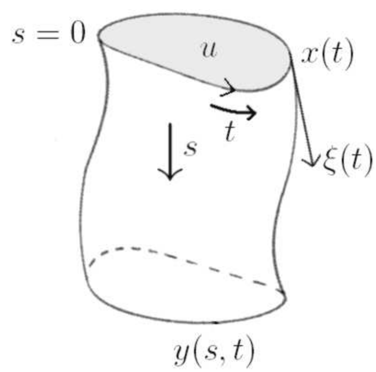
$$\int_{\mathbb{S}^2} v^* c_1 = c \int_{\mathbb{S}^2} v^*\omega$$

for any $v : \mathbb{S}^2 \to M$, where c is a positive constant. The weakly monotone case was proved by Hofer-Salamon [17] and Ono [26]. The general case was proved by Fukaya-Ono [14], Liu-Tian [21] and Ruan [29].

2. *In the monotone case, by rescaling ω if necessary, $\int_{\mathbb{S}^2} v^*\omega \in \mathbb{Z}$ for any $v : \mathbb{S}^1 \to M$. The argument in remark 2.9 shows that A is a well defined circle-valued function.*

$\mathcal{L}$ is a very large space and is not a finite dimensional manifold (except when M is a point). For $x \in \mathcal{L}$, we define a "tangent vector" ξ to be a vector field on x, i.e. $\xi(t) \in T_{x(t)}M$. In other words, ξ is a section of the induced bundle x^*TM. Fix ξ, let $y_s = y(s, \cdot)$ be a one-parameter variation of contractible loop such that $\frac{\partial y}{\partial s}(0, t) = \xi(t)$

and $y(0,t) = x(t)$. Explicitly, we can choose $y(s,t) = \exp_{x(t)}(s \cdot \xi)$ (M is a compact Riemannian manifold once J is chosen). Let $u : \mathbb{D} \to M$ be an extension of x in the sense that $u(e^{i2\pi t}) = x(t)$. Note that we can extend $y(s,\cdot)$ by the "gluing" map $y|_{\mathbb{S}^1\times[0,s]}\#u$ for each s.



Then

$$\begin{aligned}
dA(x)\xi &= \frac{d}{ds}\bigg|_{s=0}\left(-\int_{\mathbb{D}} u^*\omega - \int_0^s\int_0^1 y^*\omega + \int_0^1 H_t(y(s,t))dt\right) \\
&= \frac{d}{ds}\bigg|_{s=0}\left(\int_0^s\int_0^1 \omega(\frac{\partial y}{\partial t},\frac{\partial y}{\partial s})dt\, ds + \int_0^1 H_t(y(s,t))dt\right) \\
&= \int_0^1 (\omega(\frac{dx}{dt},\xi) + dH_t(\xi))dt.
\end{aligned}$$

Therefore $dA(x) = 0 \Leftrightarrow \omega(\dot{x},\cdot) = -dH_t$. i.e. $\dot{x}(t) = X_t(x(t))$. So critical points of the action functional correspond to the contractible periodic solutions to the Hamiltonian equation. Define

$$\mathcal{J} := \{\omega\text{-compatible almost complex structure on } M\}$$

and let $J \in \mathcal{J}$. Let g be the induced Riemannian metric. Let $\xi,\eta \in T_x\mathcal{L}$, so $\xi(t),\eta(t) \in T_{x(t)}M$. We define a Riemannian metric on $\mathcal{L}$ by

$$\langle \xi, \eta \rangle := \int_0^1 g(\xi(t), \eta(t))dt.$$

Then

$$\begin{aligned}\langle \text{grad } A, \xi \rangle &= dA(\xi) \\ &= \int_0^1 (\omega(\dot{x}, \xi) + dH_t(\xi))dt \\ &= \int_0^1 (\omega(J\dot{x}, J\xi) + \langle \nabla H_t, \xi \rangle)dt \text{ } (\nabla \text{ is the gradient w.r.t. } g) \\ &= \int_0^1 \langle J\dot{x} + \nabla H_t, \xi \rangle dt.\end{aligned}$$

So

$$\text{grad } A(x)(t) = J\dot{x}(t) + \nabla H_t(x(t)). \tag{2.6}$$

A negative gradient flow line of A is $u : \mathbb{R} \to \mathcal{L}$, $s \mapsto u_s(\cdot)$ such that

$$\frac{du}{ds} = -\text{grad } A(u_s).$$

By the above calculations, regarding $u = u(s,t) := u_s(t) : \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \to M$, u is given by the partial differential equation

$$\begin{aligned}\frac{\partial u}{\partial s} &= -J\frac{\partial u}{\partial t} - \nabla H_t(u) \\ \text{i.e. } \frac{\partial u}{\partial s} &+ J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0. \end{aligned} \tag{2.7}$$

We denote the left hand side of the above equation by

$$\overline{\partial}(u) = \overline{\partial}_{H,J}(u) := \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u).$$

Remark 2.12. *1. The equation (2.7) can also be written as*

$$\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_t) = 0.$$

This is because $J\nabla H_t = X_t$ *as* $-dH_t = \langle -\nabla H_t, \cdot\rangle = \omega(-\nabla H_t, J\cdot) = \omega(J\nabla H_t, \cdot) = \omega(X_t, \cdot)$.

2. *If* $u(s,t) \equiv x(t)$ *satisfying (2.7) is independent of* s*, then* $x(t)$ *is a critical point of* A *as grad* $A(x) = -\frac{\partial u}{\partial s} = 0$*, thus it is a periodic solution for (2.1).*

   If $H_t \equiv \text{constant}$*, then (2.7) becomes*

   $$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0,$$

   that means u *is a J-holomorphic curve. (A J-holomorphic curve is a map* u *from a Riemann surface* (Σ, i) *to an almost complex manifold* (M, J) *such that* $J \circ du = du \circ i$*, see [15])*

3. *Finally if* $H(t,x) = H(x)$ *is independent of* t*, then for those solutions* $u = u(s)$ *to (2.7) which is also independent of* t *satisfies*

   $$\frac{du}{ds} = -\nabla H(u). \tag{2.8}$$

   i.e. it satisfies the gradient flow equation for H*. This observation will be useful to relate the Morse homology with the Floer homology as we will see later in this section (see also section 4.5).*

We are going to apply Morse-type theory to study the gradient flow line of A and more importantly, understand how the behavior of the critical points of A relates with the topology of M. However there are some problems preventing us from directly applying the classical Morse theory to the action functional.

In finite dimensional Morse theory, every gradient flow line of a Morse function f on a compact manifold M "begins" and "ends" at a critical point. More precisely, if $\gamma(t)$ is a gradient flow line then $\lim_{t\to\pm\infty}\gamma(t)$ exists and the two limits are both critical points. However this is not true for the symplectic Floer theory. Actually this is true only when u is bounded.

Definition 2.13. *Let $u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$. The **energy** of u is defined by*

$$E(u) := \frac{1}{2}\int_0^1\int_{-\infty}^{\infty}(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_t(u)\right|^2)ds\ dt.$$

*u is said to be **bounded** if $E(u) < \infty$.*

Theorem 2.14. *Suppose $u = u(s,t) \in C^\infty(\mathbb{R}\times(\mathbb{R}/\mathbb{Z}), M)$ is a contractible solution of (2.7). Then $E(u) < \infty$ if and only if there exists $x^\pm \in P(H)$ such that*

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t), \tag{2.9}$$

the limits being uniform in t.

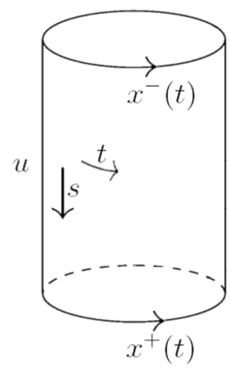


Figure 2.1: Flow line of symplectic action.

Proof. We prove "$\Leftarrow$". For u satisfying (2.9),

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_0^1\int_{-\infty}^{\infty}(\left|\frac{\partial u}{\partial s}\right|^2+\left|\frac{\partial u}{\partial t}-X_t(u)\right|^2)ds\,dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1(\left|\frac{\partial u}{\partial s}\right|^2+\left|J\frac{\partial u}{\partial t}+\nabla H_t(u)\right|^2)dt\,ds \\
&= \int_{-\infty}^{\infty}\int_0^1\left|\frac{\partial u}{\partial s}\right|^2 dt\,ds \\
&= \int_{-\infty}^{\infty}\left|\left|\frac{du}{ds}\right|\right|^2 ds \quad \text{where } ||\cdot|| \text{ is the norm in } \mathcal{L} \\
&= \int_{-\infty}^{\infty}\langle\frac{du}{ds},-\text{grad } A\rangle ds \\
&= \int_{-\infty}^{\infty}\frac{d}{ds}(-A(u_s))ds \\
&= A(x^-)-A(x^+)<\infty. \qquad (2.10)
\end{aligned}$$

We will prove the converse in theorem 4.2. $\square$

Suppose u solves (2.7) and $E(u)<\infty$, then u is called a **bounded solution** of (2.7) and denote the space of all bounded solutions of (2.7) by $\mathcal{M}$. Given $x^{\pm}\in P(H)$, we also define

Definition 2.15. *The* ***moduli space of bounded solutions***

$$\begin{aligned}\mathcal{M}(x^-,x^+) &= \mathcal{M}(x^-,x^+;H,J)\\ &:= \{u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M) \mid u \text{ solves } (2.7) \text{ and } \lim_{s\to\pm\infty} u(s,t) = x^\pm(t)\}\\ &= \{u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M) \mid \overline{\partial}(u) = 0 \text{ and } \lim_{s\to\pm\infty} u(s,t) = x^\pm(t)\}.\end{aligned}$$

In view of theorem 2.14, it is clear that

$$\mathcal{M} = \bigcup_{x,y\in P(H)} \mathcal{M}(x,y).$$

Theorem 2.14 suggests that suggests that instead of considering the flow of A on the loop space $\mathcal{L}$, one should look at the space of bounded energy solutions of its gradient flow equation.

Remark 2.16. *If $u \in \mathcal{M}(x^-,x^+)$ with $E(u) = 0$, then $\frac{\partial u}{\partial s} \equiv 0$ and so $u = u(t)$ satisfies $\frac{du}{dt} = X_t(u)$, therefore $u(t) = x^-(t) = x^+(t) \in P(H)$. In particular if $x^- \neq x^+$ and $\mathcal{M}(x^-,x^+) \neq \phi$ then the proof of theorem 2.14 shows that $A(x^-) > A(x^+)$. This is analogous to the fact in classical Morse theory that the value of a Morse function f on M decreases strictly along a non-constant gradient flow line.*

Remark 2.17. *$\mathcal{M}(x^-,x^+)$ minimizes E among all curves with bound-*

ary conditions (2.9)*, this follows from*

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 (\left|\frac{\partial u}{\partial s}\right|^2 + \left|J\frac{\partial u}{\partial t} + \nabla H\right|^2) dt\, ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 (\left|\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H\right|^2 - 2\langle\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial t} + \nabla H\rangle) dt\, ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H\right|^2 dt\, ds + \int_{-\infty}^{\infty}\langle\frac{du}{ds}, -grad\ A \circ u\rangle) ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H\right|^2 dt\, ds + A(x^-) - A(x^+).
\end{aligned}$$

There is a natural $\mathbb{R}$-action on $\mathcal{M}(x^-, x^+)$ given by $r \cdot u(s,t) = u(r+s, t)$ for $r \in \mathbb{R}$. Define the **moduli space of unparametrized bounded solutions**

$$\hat{\mathcal{M}}(x^-, x^+) := \mathcal{M}(x^-, x^+)/\mathbb{R}.$$

We study $\mathcal{M}(x, y; H, J)$ locally by linearizing $\overline{\partial}$ at $u \in \mathcal{M}(x, y)$ to get a differential operator $F(u)$. More precisely, differentiating equation (2.7) in the direction of a vector field $\xi \in C^\infty(u^*TM)$ on u leads to the first order linear differential operator $F(u) = D\overline{\partial}(u)$:

$$F(u)\xi = \nabla_s\xi + J(u)\nabla_t\xi + (\nabla_\xi J(u))\frac{\partial u}{\partial t} + \nabla_\xi \nabla H_t(u)$$

where ∇ denotes the covariant derivative with respect to the Riemannian metric given by (2.4). It turns out that if x, y are non-degenerate then $F(u)$ is a Fredholm operator with a finite index between two appropriate Sobolev spaces. We want to smoothly extend $\overline{\partial}$ as a section of a Banach bundle over a Banach manifold, so that $\mathcal{M}(x^-, x^+)$ is the zero section of $\overline{\partial}$ and argue that 0 is a regular value of $\overline{\partial}$ if we choose a suitable J. After this has been done we can apply implicit function theorem to conclude that $\mathcal{M}(x^-, x^+)$ is a smooth manifold with the dimension near u given by the index of $F(u)$.

Definition 2.18. *(H, J) is called a* ***regular pair*** *if*

1. *All contractible $x \in P(H)$ are non-degenerate, and*
2. *If $x^\pm \in P(H)$ are contractible and $u \in \mathcal{M}(x^-, x^+)$, then $F(u)$ is surjective.*

Due to an infinite dimensional version of Sard's theorem by Smale [35] (see also [13], [33] for the details), we have a transversality result:

Proposition 2.19. *There is a dense subset of smooth almost complex structure $\mathcal{J}_{\mathrm{reg}} \subset C^\infty(\mathrm{End(TM)})$ such that for all $J \in \mathcal{J}_{\mathrm{reg}}$ and $u \in \mathcal{M}$, $F(u)$ is onto, i.e. (H, J) is regular.*

So by implicit function theorem and by choosing a regular pair, we have

Theorem 2.20. *For a regular pair (H, J), $\mathcal{M}(x^-, x^+; H, J)$ is a finite dimensional smooth manifold for $x^\pm \in P(H)$ and the dimension of $\mathcal{M}(x^-, x^+)$ is given by the index:*

$$\dim \mathcal{M}(x^-, x^+) = \operatorname{index} F(u).$$

We will see that $\operatorname{index} F(u)$ can be in turn calculated by the difference of a Maslov-type index μ:

$$\operatorname{index} F(u) = \mu(x^-) - \mu(x^+).$$

The Maslov index $\mu : P(H) \to \mathbb{Z}$ associates an integer to every contractible periodic solution $x \in P(H)$ of (2.1). The Maslov index will play the role of grading the Floer homology groups just like the Morse index does in Morse homology.

2.4 Construction of Floer Homology

It follows from the manifold structure of $\mathcal{M}(x,^-, x^+)$ and a compactness-gluing argument using the Gromov's compactness theorem [15] that we have the following theorem.

Proposition 2.21. *Suppose (H, J) is regular, $x^\pm \in P(H)$ such that $\mu(x^-) - \mu(x^+) = 1$. Then $\hat{\mathcal{M}}(x^-, x^+)$ is a compact 0-dimensional manifold, i.e. it consists of finite number of points. In other words the set of connecting trajectories (modulo shifting) between x^- and x^+ is finite.*

This result is used to construct the boundary operator in Floer homology. The so called compactness-gluing argument is very useful and it is used repeatedly to prove several results as we will see in chapter 4. Let (H, J) be a fixed regular pair. For simplicity, we work with $\mathbb{Z}_2$ coefficient only.

Definition 2.22. *Define the* ***k-th Floer chain complex*** *as the vector space over* $\mathbb{Z}_2$ *generated by the periodic solution* $x \in P(H)$ *of* (2.1) *with Maslov index* k

$$C_k := span_{\mathbb{Z}_2}\{x \in P(H) : \mu(x) = k\}.$$

If $\mu(x) - \mu(y) = 1,$ *define*

$$\langle \partial x, y \rangle := \#\hat{\mathcal{M}}(x,y) \quad (mod\ 2).$$

The boundary operator $\partial_k : C_k \to C_{k-1}$ *is defined as*

$$\partial_k x := \sum_{y \in C_{k-1}} \langle \partial x, y \rangle y \quad for\ x \in C_k$$

and extends it linearly. (C_*, ∂_*) *is called the* ***Floer chain complex****.*

By analyzing an appropriate moduli space using the compactness-gluing argument again, Floer proved the following in the monotone case, establishing the existence of Floer homology:

Theorem 2.23 (Floer [11])**.**

$$\partial_k \circ \partial_{k+1} = 0.$$

Definition 2.24. *Define the* ***Floer homology groups*** *of the pair* (H, J) *on* (M, ω)

$$HF_k(M; H, J) := \ker \partial_k / \mathrm{im}\ \partial_{k+1}.$$

It is remarkable that these homology groups turn out to be independent of the choice of the pair (H, J). One important theorem is

Theorem 2.25 (Floer continuation [11])**.** *Suppose* $(H^\alpha, J^\alpha), (H^\beta, J^\beta)$ *are two regular pairs on* M*. Then there exists a natural isomorphism*

$$\phi^{\beta\alpha} : HF_*(M; H^\alpha, J^\alpha) \to HF_*(M; H^\beta, J^\beta).$$

Furthermore, if (H^γ, J^γ) *is another regular pair, then*

$$\phi^{\alpha\beta} \circ \phi^{\beta\gamma} = \phi^{\alpha\gamma} \text{ and } \phi^{\alpha\alpha} = \mathrm{id}.$$

With the observation that if $H(t, x) = H(x)$ is independent of t, then the gradient flow line $u = u(s)$ of H:

$$\frac{du}{ds} = -\nabla H(u). \tag{2.11}$$

actually solves the P.D.E. (2.7). This makes a relation to Morse theory. In fact we can find a sufficiently C^2 small Morse function H independent of t and an almost complex structure $J \in \mathcal{J}$ such that (H, J) is regular and every bounded solution u of (2.7) is independent of t. Then in this case all $x \in P(H)$ are exactly the critical points of H, furthermore the Maslov index of $x \in P(H)$ agrees with the Morse index of x regarded as a critical point up to a shifting of n, so the Floer's complex of (H, J) agrees with the Morse complex of the gradient flow (2.11). Therefore

$$HF_k(M; H, J) \cong HM_{n+k}(M; H, J).$$

Note that until now we can deduce something about the existence of periodic solutions to (2.1). Finally by the Morse homology theorem

(theorem 1.24) the Morse homology is isomorphic to the singular homology of M:

$$HM_k(M; H, J) \cong H_k(M; \mathbb{Z}_2).$$

So combining all these we have

Theorem 2.26 (Floer[11])**.** *The Floer homology is isomorphic to the singular homology of M up to a shift of grading:*

$$HF_k(M; H, J) \cong H_{k+n}(M; \mathbb{Z}_2), \quad -n \le k \le n.$$

In particular by the weak Morse inequalities (theorem 1.25), Floer [11] proved the Arnold conjecture as a corollary.

Theorem 2.27 (Arnold conjecture)**.** *If all the periodic solutions to* (2.1) *are non-degenerate, then*

$$\#P(H) \ge \sum_{i=0}^{2n} b_i.$$

Chapter 3

Fredholm Theory

We would like to study the moduli space $\mathcal{M}(x^-, x^+)$, which turns out to be a smooth finite dimensional manifold for a generic choice of the pair (H, J). This space can be analyzed locally by covariant differentiating equation (2.7) in the direction of a smooth vector field $\xi \in C^\infty(u^*TM)$ on $u \in \mathcal{M}(x^-, x^+)$. This leads to the first order linear differential operator $F(u) = D\overline{\partial}(u)$:

$$F(u)\xi = \nabla_s\xi + J(u)\nabla_t\xi + (\nabla_\xi J(u))\frac{\partial u}{\partial t} + \nabla_\xi \nabla H_t(u) \qquad (3.1)$$

where ∇ denotes the covariant derivative with respect to the metric induced by some almost complex structure $J \in \mathcal{J}$. This operator in turn is studied by Fredholm theory. However, in order to use Fredholm theory, we cannot work only on the space $C^\infty(u^*TM)$ of smooth sections of u^*TM as this space is not a Banach space. We need to find an alternative functional setting.

There are several goals to achieve in this chapter. Firstly we have to find the right analytical setup: we want to regard $\overline{\partial}$ as a section of a Banach bundle over a certain Banach manifold, so that we can identify $\mathcal{M}(x^-, x^+)$ as the zero of this section. Secondly we want to show that $\overline{\partial}$ is Fredholm by linearizing it at $u \in \mathcal{M}(x^-, x^+)$ (we can differentiate $\overline{\partial}$ at u without specifying a connection because u is a zero of this section) and prove that the linearized operator $F(u)$ is a linear Fredholm operator, the surjectivity of $F(u)$ means that the section $\overline{\partial}$ intersects with the zero section transversely. After this has been done, we can then apply the implicit function theorem for Banach manifolds to show that $\mathcal{M}(x^-, x^+)$ is a finite dimensional smooth manifold if in addition, 0 is a regular value of $\overline{\partial}$. This not always true, but is true for a generic choice of H and J, as we will see in the chapter 4. In addition to proving $\overline{\partial}$ is Fredholm, we also get a formula for the index of $\overline{\partial}$ in terms of the Maslov index of $x^{\pm}$, so that we can calculate the dimension of $\mathcal{M}(x^-, x^+)$ by calculating the Maslov index of $x^{\pm}$, i.e. it depends on its endpoints only.

3.1 Fredholm Operator

Definition 3.1. *Let X, Y be Banach spaces. A bounded linear operator $F : X \to Y$ is a* ***Fredholm operator*** *if it has a finite dimensional kernel and cokernel and it has a closed range.*
The (Fredholm) index of F is defined by

$$\operatorname{index} F := \dim\ker F - \dim \operatorname{coker} F = \dim\ker F - \operatorname{codim} \mathrm{R}(F)$$

where $R(F)$ denotes the range of F.

The set of Fredholm operators $F(X,Y)$ from X to Y in open in the set $L(X,Y)$ of bounded linear operators with norm topology. The index is continuous on $F(X,Y)$. Furthermore, Fredholm operator and its index is stable under compact perturbation. This means that if $F : X \to Y$ is Fredholm and $K : X \to Y$ is a compact linear operator, then $F+K$ is also Fredholm and $\mathrm{index}\,(F+K) = \mathrm{index}\,F$. The nonlinear extension of the above notion fits in the context of smooth manifolds, or Banach manifolds, which can be regarded as a smooth manifold in which each point has a neighborhood diffeomorphic to an open set in a Banach space, which can be infinite dimensional (see [20]). Let M, N be two connected manifolds. A C^1 map $f : M \to N$ is Fredholm if for each $x \in M$, the differential $df(x) : T_xM \to T_{f(x)}N$ is Fredholm. The Fredholm index of f is defined as the index of $df(s)$. Since df is continuous and M is connected the index is independent of x by the above remark.

3.2 The Linearized Operator

It was Floer who found the appropriate functional setting to set up his theory of Floer homology.

Let $x^{\pm} \in P(H)$ be a pair of non-degenerate solutions of (2.1) and

$u : \mathbb{R} \times \mathbb{S}^1 \to M$ be smooth such that

$$\lim_{s \to \pm\infty} u(s,t) \quad = \quad x^{\pm}(t), \tag{3.2}$$

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial t} = \dot{x}^{\pm}(t) \quad \text{and} \quad \lim_{s \to \pm\infty} \frac{\partial u}{\partial s} = 0 \tag{3.3}$$

where all limits are uniform in t. For such u, and for a smooth compactly supported vector field $\xi \in C^{\infty}(u^*TM)$, i.e. $\xi(s,t) \in T_{u(s,t)}M$, the L^p norm of ξ is given by

$$||\xi||_{L^p} = (\int_{-\infty}^{\infty} \int_0^1 |\xi(s,t)|^p dt\ ds)^{\frac{1}{p}}.$$

We define the Hilbert space $L^p(u^*TM)$ as the completion of all smooth compactly supported vector fields on u^*TM with respect to this norm. We also define

$$W^{1,p}(u^*TM) := \{\xi \in L^p(u^*TM) \mid \nabla_s \xi, \nabla_t \xi \in L^p(u^*TM)\}$$

where $\nabla_s \xi$, $\nabla_t \xi$ denotes the covariant derivative of ξ with respect to s and t respectively in the weak sense. Now let $\xi \in W^{1,p}(u^*TM)$ and define $\exp \xi : \mathbb{R} \times \mathbb{S}^1 \to M$ by $\exp \xi(s,t) := \exp_{u(s,t)} \xi(s,t)$.
Define

$$\begin{aligned} \mathcal{P}^{1,p} &= \mathcal{P}^{1,p}(x^-, x^+) \\ &:= \{\exp \xi \mid u \in C^{\infty}(\mathbb{R} \times \mathbb{S}^1, M) \text{ with } (3.2), (3.3), \xi \in W^{1,p}(u^*TM)\}. \end{aligned}$$

Then it can be proved that $\mathcal{P}^{1,p}$ is a Banach manifold with tangent space (see [34])

$$T\mathcal{P}^{1,p} = \bigcup_{u \in \mathcal{P}^{1,p}} W^{1,p}(u^*TM).$$

This is a Banach bundle on $\mathcal{P}^{1,p}$ with characteristic fiber $W^{1,p}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$. Analogously we define the Banach bundle

$$L^p(\mathcal{P}^{1,p*}TM) := \bigcup_{u \in \mathcal{P}^{1,p}} L^p(u^*TM)$$

over $\mathcal{P}^{1,p}$ with characteristic fiber $L^p(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$. Then in this setting we can extend $\overline{\partial}$ as a smooth section $\overline{\partial} : \mathcal{P}^{1,p} \to L^p(\mathcal{P}^{1,p})$ from $\mathcal{P}^{1,p}$ into the bundle $L^p(\mathcal{P}^{1,p})$ for $p > 2$. For $u \in \mathcal{P}^{1,p}$ with $\overline{\partial}(u) = 0$, the linear differential operator $F(u) : W^{1,p}(u^*TM) \to L^p(u^*TM)$ defined by

$$F(u)\xi := \nabla_s \xi + J(u)\nabla_t \xi + (\nabla_\xi J(u))\frac{\partial u}{\partial t} + \nabla_\xi \nabla H_t(u) \qquad (3.4)$$

can be viewed as the differential of $\overline{\partial}$ at $u \in \mathcal{P}^{1,p}$. We will show that $F(u)$ is Fredholm.

3.3 Maslov Index

Recall that the group of $2n \times 2n$ symplectic matrices $Sp(2n, \mathbb{R}) := \{A \in M(2n, \mathbb{R}) : A^T J_0 A = J_0\}$, where

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $I \in M(n, \mathbb{R})$ is the $n \times n$ identity matrix. There is a natural embedding of the unitary group $U(n)$ into $Sp(2n, \mathbb{R})$, namely $\iota : U(n) \to Sp(2n, \mathbb{R})$ given by

$$\iota : X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

This embedding is in fact a group homomorphism and we identify $U(n)$ with its image in $Sp(2n, \mathbb{R})$. Salamon and Zehnder showed that there is a natural continuous extension ρ of the determinant map $\det : U(n) \to \mathbb{S}^1$ with some natural properties. However this is no longer a homomorphism. The general setting is as follows.

Let V be a finite dimensional real symplectic vector space. That means V is equipped with a non-degenerate skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. Let $Sp(V, \omega) = \{A \in L(V) : A^*\omega = \omega\}$. Then

Proposition 3.2 ([5], [32]). *For each (V, ω) as above, there is a unique continuous map $\rho = \rho_V : Sp(V, \omega) \to \mathbb{S}^1$ satisfying the following conditions:*

1. *(Naturality). If $T : (V_1, \omega_1) \to (V_2, \omega_2)$ is a symplectic isomorphism and $A \in Sp(2n, \mathbb{R})$, then $\rho(TAT^{-1}) = \rho(A)$.*

2. *(Product). If $(V, \omega) = (V_1 \oplus V_2, \omega_1 \oplus \omega_2)$, then $\rho(A_1 \oplus A_2) = \rho(A_1)\rho(A_2)$ where $A_1 \oplus A_2 \in Sp(V, \omega)$ is given by $A_1 \oplus A_2(z_1, z_2) = (A_1 z_1, A_2 z_2), A_i \in Sp(V_i, \omega_i), z_i \in V_i$.*

3. *(Determinant). If $A \in Sp(2n, \mathbb{R}) \cap U(n)$ is given by $A = \iota(X + iY)$, then $\rho(A) = \det(X + iY)$.*

4. *(Normalization). If A has no eigenvalue of the form $e^{i\theta} \in \mathbb{S}^1$, then $\rho(A) = \pm 1$.*

Note that we do not differentiate different ρ when the domain of definition is clear. We now turn into the Maslov index for symplectic paths.

Definition 3.3. $\mathcal{LP}^* := \{\Psi : [0,1] \to Sp(2n,\mathbb{R}) : \Psi(0) = I, \det(I - \Psi(1)) \neq 0\}$. *Such paths are called* ***non-degenerate paths****.*

For a curve $\Psi : [0,1] \to Sp(2n,\mathbb{R})$, there exists a lifting $\alpha : [0,1] \to \mathbb{R}$ such that $\rho \circ \Psi(t) = e^{i\pi\alpha(t)}$, define $\Delta(\Psi) := \alpha(1) - \alpha(0)$, clearly this is independent of the choice of α.

Definition 3.4. $Sp^* = Sp(2n,\mathbb{R})^* := \{A \in Sp(2n,\mathbb{R}) : \det(A - I) \neq 0)\}$.

Lemma 3.5 ([33])**.** *Sp^* has two path-connected components*

$$Sp^{\pm} = Sp(2n,\mathbb{R})^{\pm} := \{A \in Sp(2n,\mathbb{R}) : \pm\det(A - I) > 0\}.$$

Moreover, every loop in Sp^* is contractible in $Sp(2n,\mathbb{R})$.

Let $W^+ = -I$ and $W^- = \mathrm{diag}(2, -1, \cdots, -1, 1/2, -1, \cdots, -1)$. It can be proved that $W^{\pm} \in Sp^{\pm}$. By the above lemma, for $\Psi \in \mathcal{LP}^*$ so that $A := \Psi(1) \in Sp^*$, there exists $\bar{\Psi} : [0,1] \to Sp^*$ such that $\bar{\Psi}(0) = A$, $\bar{\Psi}(1) \in \{W^+, W^-\}$.

By lemma 3.5, for fixed $A \in Sp^*$, $\Delta(\bar{\Psi})$ is independent of choice of $\bar{\Psi}$, for if we choose two such paths $\bar{\Psi}_1, \bar{\Psi}_2$, then they are homotopic in $Sp(2n,\mathbb{R})$ relative to their endpoints and thus can be lifted to $\mathbb{R}$ with the same endpoints.

Definition 3.6. *For $\Psi \in \mathcal{LP}^*$, the **Maslov index** of Ψ is defined by*

$$\mu(\Psi) := \Delta(\Psi) + \Delta(\bar{\Psi}).$$

Proposition 3.7.

1. $\mu(\Psi) \in \mathbb{Z}$.

2. $\mu(\Psi_0) = \mu(\Psi_1) \Leftrightarrow \Psi_0$ *and* Ψ_1 *are homotopic in* $\mathcal{LP}^*$.

3. *Suppose* $\Psi(t) = \exp(J_0 St)$ *where* $S = S^T \in M(2n, \mathbb{R})$ *is a non-singular symmetric real matrix with norm* $|S| < 2\pi$. *Then* $\Psi \in \mathcal{LP}^*$ *and*

$$\mu(\Psi) = \lambda(S) - n$$

where $\lambda(S)$ *denotes the number of negative eigenvalues of* S *counted with multiplicity.*

Proof. We will prove (3). Choose a path $P_\tau \in SO(2n)$ such that $P_0 = I$ and $S_1 := P_1^T SP$ is a diagonal matrix, $\tau \in [0,1]$. Then $S_\tau := P_\tau^T SP_\tau$ is a symmetric path joining $S = S_0$ to a diagonal matrix D and $\lambda(S_\tau)$ is independent of τ.

Now for $|S| = |D| < 2\pi$, we have $|J_0 Dx| \leq |D||x| < 2\pi|x|$ for $x \neq 0 \Rightarrow 2\pi i$ is not an eigenvalue of $J_0 D$.

Then $\sigma(\exp(J_0 S_\tau)) = \sigma(\exp(J_0 D)) = \{e^\mu : \mu \in \sigma(J_0 D)\} \Rightarrow 1 \notin \sigma(\exp(J_0 S_\tau))$. This implies $\Psi_\tau \in \mathcal{LP}^*$ for all τ where $\Psi_\tau(t) := \exp(J_0 S_\tau t)$.

So we can assume S is diagonal. Without loss of generality we assume that

$$S = \mathrm{diag}(\varepsilon, \cdots, \varepsilon, -\varepsilon, \cdots, -\varepsilon)$$

where $0 < \varepsilon < 2\pi$ and there are $k = \lambda(S)$ many $-\varepsilon$'s.

Decompose S into 2×2 blocks, with each block equals to one of the following:

$$S_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, S_1 = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, S_2 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}.$$

These are the case in (iii) for $n = 1$.

$$\exp(J_0 S_0 t) = \begin{pmatrix} \cos(\varepsilon t) & \sin(\varepsilon t) \\ -\sin(\varepsilon t) & \cos(\varepsilon t) \end{pmatrix} = \iota(e^{-i\varepsilon t}).$$

By the determinant property in proposition 3.2, $\rho(\exp(J_0 S_0 t)) = e^{-i\varepsilon t}$. Since $0 < \varepsilon < 2\pi$, $\mu(\Psi) = -1 = \lambda(S_0) - 1$ in this case.

Similarly, for $S = S_1$, $\mu(\Psi) = 0 = \lambda(S) - 1$ and $\mu(\Psi) = 1$ for $S = S_2$. All together, this implies

$$\mu(\Psi) = \lambda(S) - n$$

in general. □

Proposition 3.8. *Let $\bar{x} : \mathbb{D} \to M$ be smooth. Then there exists a (unitary) trivialization $\Phi : \mathbb{D} \times \mathbb{R}^{2n} \to \bar{x}^*TM$: $(z, \xi) \mapsto \Phi(z)\xi$ such that*

$$\Phi J_0 = J\Phi,\ \omega(\Phi\xi, \Phi\eta) = \omega_0(\xi, \eta) \ \text{ and } \ g(\Phi\xi, \Phi\eta) = \xi^T \eta$$

where J_0, ω_0 are the standard almost complex structure and the standard symplectic structure in $\mathbb{R}^{2n}$ respectively. Two such trivializations are homotopic.

Proof. Since $\mathbb{D}$ is contractible, by choosing a complex trivialization of $\bar{x}^*TM$ as complex vector bundle and applying Gram-Schmidt process, we obtain a smooth complex orthonormal frames $\{v_1, \cdots, v_n\}$ on $\bar{x}^*TM$. Then define $\Phi(e_i) = v_i$, $\Phi(e_{n+i}) = Jv_i$ for $i = 1, \cdots, n$. Suppose now Φ_1, Φ_2 are two such trivializations. Then $\Phi_2^{-1}\Phi_1 : \mathbb{D} \to U(n)$ is homotopic to the constant map $z \mapsto I$ as $U(n)$ is path connected. This shows that Φ_1 and Φ_2 are homotopic. $\square$

Let $x \in C^\infty(\mathbb{S}^1, M)$ be a contractible solution to (2.1):

$$\dot{x} = X_t(x)$$

so that there exists $\bar{x} : \mathbb{D} \to M$ with $\bar{x}(e^{i2\pi t}) = x(t)$. Then by proposition 3.8, there exists a symplectic orthogonal trivialization of $\bar{x}^*TM$, which induces

$$\Phi(t) = \Phi(t+1) : \mathbb{R}^{2n} \to T_{x(t)}M$$

a trivialization of x^*TM.

Lemma 3.9. *If* $c_1(\pi_2(M)) = 0$ *(assumption* (2.5)*), then the trivialization* Φ *is independent of the choice of the extension* $\bar{x} : \mathbb{D} \to M$*.*

Proof. Suppose $\bar{x}, \bar{x}' : \mathbb{D} \to M$ satisfies $\bar{x}(e^{i2\pi t}) = \bar{x}'(e^{i2\pi t}) = x(t)$ which induces the two trivializations Φ, Φ' of $\bar{x}^*TM$ and $\bar{x}'^*TM$

respectively. By a small perturbation (slight "thickening" of the disk) if necessary we can assume that

$$\bar{x}(re^{i\theta}) = \bar{x}(e^{i\theta}), \Phi(re^{i\theta}) = \Phi(e^{i\theta})$$

for $1 - \varepsilon \leq r \leq 1$, and similarly for Φ' and $\bar{x}'$.

The map $u : \mathbb{S}^2 = \mathbb{C} \cup \{\infty\} \to M$ defined by

$$u(z) = \begin{cases} \Phi(z) & \text{if } |z| \leq 1 \\ \Phi'(1/\bar{z}) & \text{if } |z| \geq 1 \end{cases}$$

is smooth. As $c_1(u) = 0$ the $\mathbb{C}^n$ bundle u^*TM is trivial. Hence there exists a symplectic orthogonal trivialization $\Theta : \mathbb{S}^2 \times \mathbb{R}^{2n} \to u^*TM$. Hence by proposition 3.8, the two trivializations $\Phi(t)$ and $\Phi'(t)$ are both homotopic to $\Theta(e^{i2\pi t})$. □

Let $\Phi : [0, 1] \times \mathbb{R}^{2n} \to x^*TM$ be a symplectic orthogonal trivialization of x^*TM, i.e. $\Phi_t = \Phi(t) : \mathbb{R}^{2n} \xrightarrow{\cong} T_{x(t)}M$ for $t \in [0, 1]$. Recall that ψ_t is the flow of X_t. Define the path

$$\Psi(t) := \Phi_t^{-1} d\psi_t(x_0)\Phi_0, \quad t \in [0, 1] \tag{3.5}$$

where $d\psi_t(x_0)$ denotes the differential map of ψ_t at the point $x_0 = x(0)$. Then $\Psi(t)$ is a path into $Sp(2n, \mathbb{R})$ because $\Psi(t)^*\omega_0 = \Phi_0^*\psi_t^*(\Phi_t^{-1})^*\omega_0 = \omega_0$. Also, $\Psi(0) = I$ and $\Psi(1) \in Sp^*$ by the non-degenerate condition, i.e. $\Psi \in \mathcal{LP}^*$, clearly this condition depends on $d\psi_t$ only and is independent of the choice of Φ.

Definition 3.10. *For a contractible periodic solution $x \in P(H)$ of* (2.1)*, define the* ***Maslov index*** *of x by*

$$\mu(x) := \mu(\Psi).$$

Remark 3.11. *If we choose two trivializations Φ, Φ', by proposition 3.8, they are homotopic and so induces homotopic paths $\Psi, \Psi' \in \mathcal{LP}^*$, by proposition 3.7, Ψ and Ψ' have the same Maslov index. Therefore $\mu(x)$ is independent of choice of Φ and hence Ψ.*

3.4 Fredholm Index

Definition 3.12. *Let $\xi : \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) = \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^{2n}$ be a map. The L^p norm of ξ is defined to be*

$$||\xi||_{L^p} = ||\xi||_{L^p(\mathbb{R}\times\mathbb{S}^1,\mathbb{R}^{2n})} := (\int_{-\infty}^{\infty}\int_0^1 |\xi|^p dt\ ds)^{\frac{1}{p}}.$$

Define

$$L^p(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) := \{\xi : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^{2n} \mid ||\xi||_{L^p} < \infty\}$$

and

$$W^{1,p}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n}) := \{\xi \in L^p(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n}) \mid \frac{\partial \xi}{\partial s}, \frac{\partial \xi}{\partial t} \in L^p(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})\}$$

where the derivatives are understood to be in the weak sense. The $W^{1,p}$ norm of $\xi \in W^{1,p}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ is defined to be

$$||\xi||_{W^{1,p}} = ||\xi||_{W^{1,p}(\mathbb{R}\times\mathbb{S}^1,\mathbb{R}^{2n})} := (||\xi||^p_{L^p} + \left|\left|\frac{\partial \xi}{\partial s}\right|\right|^p_{L^p} + \left|\left|\frac{\partial \xi}{\partial t}\right|\right|^p_{L^p})^{\frac{1}{p}}.$$

Consider for $p \geq 2$, the operator $F : W^{1,p}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ defined by

$$F\xi := \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S\xi, \tag{3.6}$$

where $S^T = S = S(s,t) \in M(2n, \mathbb{R})$ is a continuous matrix valued function on $\mathbb{R} \times \mathbb{S}^1$ such that

$$S^{\pm} := \lim_{s \to \pm\infty} S(s,t)$$

exists with uniform convergence in t. When $S = 0$, F is called the Cauchy-Riemann operator. This operator is considered because we will show (see the proof of theorem 3.16) that by applying a perturbation (which does not affect the Fredholm property) if necessary, the operator $F(u)$ in (3.1) is of this form after a trivialization.

There is a one-one correspondence between S and $\Psi(s,t) \in Sp(2n, \mathbb{R})$ defined by the differential equation

$$\begin{cases} \frac{\partial \Psi}{\partial t} = J_0 S \Psi, \\ \Psi(s,0) = I_{2n \times 2n}. \end{cases} \tag{3.7}$$

$\Psi(s,t) \in Sp(2n, \mathbb{R})$ as for fixed s,

$$\frac{d}{dt}(\Psi^T J_0 \Psi) = \Psi^T S^T J_0^T J_0 \Psi + \Psi^T J_0 J_0 S \Psi = 0$$

and $\Psi(s,0) \in Sp(2n, \mathbb{R})$ for all s. It turns out that Ψ converges uniformly in t as $s \to \pm\infty$. Denote

$$\Psi^{\pm}(t) := \lim_{s \to \pm\infty} \Psi(s,t).$$

Proposition 3.13 ([33] theorem 4.1)**.** *If* $\Psi^{\pm} \in \mathcal{LP}^*$*, then*

1. *F is a Fredholm operator, and*
2. $\operatorname{index} F = \mu(\Psi^-) - \mu(\Psi^+)$.

The proof uses the following lemma (see for example [24], [34]).

Lemma 3.14. *Let X, Y, Z be Banach spaces, $F : X \to Y$ is a bounded linear operator and $K : X \to Z$ is compact. Suppose there exists $c > 0$ such that for all $\xi \in X$,*

$$||\xi||_X \leq c(||F\xi||_Y + ||K\xi||_Z).$$

Then F has a closed range and its kernel is finite dimensional. F with this property is said to be semi-Fredholm.

The Fredholm property of F is easiest to prove for $p = 2$, which will be done here. For $p > 2$, see [31]. We will assume $p = 2$ from now on.

Proof of Proposition 3.13 (1). Let $X := \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) = \mathbb{R} \times \mathbb{S}^1$. The proof of (1) consists of four steps.

Step 1.

Suppose $S(s,t) = S(t)$, $\Psi(s,t) = \Psi(t)$ are both independent of s. We claim that there exists $c > 0$ such that

$$||\xi||_{W^{1,2}(X)} \leq c||F\xi||_{L^2(X)}.$$

Define the symmetric operator $A : W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ by

$$A\xi(t) = J_0 \frac{d\xi}{dt} + S(t)\xi(t).$$

Then

$$\begin{aligned}
\ker A \neq 0 \;&\Leftrightarrow\; \exists \xi(t) \neq 0, \quad A\xi = 0 \\
&\Leftrightarrow\; \xi(t) = \Psi(t)\xi(0) \quad \text{by uniqueness of solution to (3.7)} \\
&\Leftrightarrow\; \Psi(1)\xi(0) = \xi(0) \neq 0 \text{ i.e. } 1 \in \sigma(\Psi(1)) \\
&\Leftrightarrow\; \Psi \notin \mathcal{LP}^*. \qquad (3.8)
\end{aligned}$$

Therefore A is invertible onto its range and so exists $c_0 > 0$ such that

$$||\xi||_{W^{1,2}(\mathbb{S}^1)} \leq c_0 ||A\xi||_{L^2(\mathbb{S}^1)}.$$

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and consider the Fourier transform $\mathcal{F} : L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ defined by

$$(\mathcal{F}\xi)(\omega, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\omega s)\xi(s,t)ds.$$

Then $\mathcal{F}$ is an isometry. Also denote $\mathcal{F}\xi$ by $\hat{\xi}$. Then $\mathcal{F}(\frac{\partial \xi}{\partial s}) = i\omega\hat{\xi}$ and so

$$\mathcal{F}(\frac{\partial \xi}{\partial s} + A\xi) = i\omega\hat{\xi} + A\hat{\xi}. \qquad (3.9)$$

So now

$$\begin{aligned}
||\xi||^2_{W^{1,2}(X)} &= \int_{-\infty}^{\infty} (\left|\left|\frac{\partial \xi}{\partial t}\right|\right|^2_{L^2(\mathbb{S}^1)} + \left|\left|\frac{\partial \xi}{\partial s}\right|\right|^2_{L^2(\mathbb{S}^1)} + ||\xi||^2_{L^2(\mathbb{S}^1)})ds \\
&= \int_{-\infty}^{\infty} (\left|\left|\frac{\partial \hat{\xi}}{\partial t}\right|\right|^2_{L^2(\mathbb{S}^1)} + ||i\omega\hat{\xi}||^2_{L^2(\mathbb{S}^1)} + ||\hat{\xi}||^2_{L^2(\mathbb{S}^1)})d\omega \\
&= \int_{-\infty}^{\infty} (||\hat{\xi}||_{W^{1,2}(\mathbb{S}^1)} + \omega^2 \, ||\xi||^2_{L^2(\mathbb{S}^1)})d\omega. \qquad (3.10)
\end{aligned}$$

Consider

$$||\hat{\xi}||_{W^{1,2}(\mathbb{S}^1)} \leq c_0||A\hat{\xi}||_{L^2(\mathbb{S}^1)} \leq c_0||A\hat{\xi} + i\omega\hat{\xi}||_{L^2(\mathbb{S}^1)} \tag{3.11}$$

for $\omega \in \mathbb{R}$ as A is symmetric and $||A\hat{\xi} + i\omega\hat{\xi}||^2_{L^2(\mathbb{S}^1)} = ||A\hat{\xi}||^2_{L^2(\mathbb{S}^1)} + \omega^2||\hat{\xi}||^2_{L^2(\mathbb{S}^1)}$. Also,

$$\begin{aligned} |\omega| \cdot ||\hat{\xi}||^2_{L^2(\mathbb{S}^1)} &\leq ||\langle\hat{\xi}, i\omega\hat{\xi} + A\hat{\xi}\rangle||_{L^2(\mathbb{S}^1)} \\ &\leq ||\hat{\xi}||_{L^2(\mathbb{S}^1)}||i\omega\hat{\xi} + A\hat{\xi}||_{L^2(\mathbb{S}^1)} \\ \Rightarrow |\omega| \cdot ||\hat{\xi}||_{L^2(\mathbb{S}^1)} &\leq ||i\omega\hat{\xi} + A\hat{\xi}||_{L^2(\mathbb{S}^1)}. \end{aligned} \tag{3.12}$$

So by (3.11) and (3.12), (3.10) becomes

$$||\xi||^2_{W^{1,2}(X)} \leq (c_0^2+1)\int_{-\infty}^{\infty} ||A\hat{\xi}+i\omega\hat{\xi}||_{L^2(\mathbb{S}^1)}d\omega = c||F\xi||^2_{L^2(X)} \quad \text{by (3.9)}$$

where $c = c(S)$.

Step 2.

For general S and hence F, we claim that there exists sufficiently large $T > 0$, $c = c(T) > 0$ such that for all $\xi \in W^{1,2}(X, \mathbb{R}^{2n},)$ with $\xi|_{[-T,T]} = 0$, then

$$||\xi||_{W^{1,2}(X)} \leq c||F\xi||_{L^2(X)}.$$

By step 1, there exists $c^{\pm}$ such that

$$||\xi||_{W^{1,2}(X)} \leq c^{\pm}||F^{\pm}\xi||_{L^2(X)}$$

for the limit operators $F^{\pm}\xi = \frac{\partial\xi}{\partial s} + J_0\frac{\partial\xi}{\partial t} + S^{\pm}\xi$.

Let $c = \max(c^+, c^-)$ and let $\varepsilon > 0$. Then there exists $T > 0$ such that for all t,

$$\begin{cases} |S(s,t) - S^-(t)| < \varepsilon & \text{for } s \leq -T, \\ |S(s,t) - S^+(t)| < \varepsilon & \text{for } s \geq T. \end{cases}$$

For $\xi|_{[-T,T]} = 0$, $\xi = \xi^- + \xi^+$ where $\begin{cases} \xi^-|_{[-T,\infty]} = 0, \\ \xi^+|_{[-\infty,T]} = 0. \end{cases}$

$$\begin{aligned} ||F^\pm \xi^\pm||_{L^2(X)} &\leq ||(F^\pm - F)\xi^\pm||_{L^2(X)} + ||F\xi^\pm||_{L^2(X)} \\ &\leq \varepsilon ||\xi^\pm||_{L^2(X)} + ||F\xi^\pm||_{L^2(X)} \quad \text{as } F^\pm - F = S^\pm \end{aligned}$$

$$\begin{aligned} \therefore ||\xi||_{W^{1,2}(X)} &= ||\xi^-||_{W^{1,2}(X)} + ||\xi^+||_{W^{1,2}(X)} \\ &\leq c(||F^-\xi^-||_{L^2(X)} + ||F^+\xi^+||_{L^2(X)}) \\ &\leq c(||F\xi^-||_{L^2(X)} + ||F\xi^+||_{L^2(X)} + \varepsilon(||\xi^-||_{L^2(X)} + ||\xi^+||_{L^2(X)})) \\ &= c(||F\xi||_{L^2(X)} + \varepsilon||\xi||_{L^2(X)}) \\ &\leq c(||F\xi||_{L^2(X)} + \varepsilon||\xi||_{W^{1,2}(X)}) \end{aligned}$$

$$\therefore (1 - c\varepsilon)||\xi||_{W^{1,2}(X)} \leq c||F\xi||_{L^2(X)}.$$

Take $\varepsilon < 1/c$ and for the corresponding $T = T(\varepsilon)$, then if $\xi|_{[-T,T]} = 0$,

$$||\xi||_{W^{1,2}(X)} \leq \frac{c}{1 - c\varepsilon}||F\xi||_{L^2(X)} = c'||F\xi||_{L^2(X)}.$$

Step 3.

Claim: there exists $c > 0$, a Banach space Z, a compact operator $K : X \to Z$ such that

$$|||\xi||_{W^{1,2}(X)} \leq c(||F\xi||_{L^2(X)} + ||K\xi||_Z). \tag{3.13}$$

This estimate shows that F is semi-Fredholm by lemma 3.14. Now let $T = T(S)$ be given by step 2. Let $\xi \in W^{1,2}(X, \mathbb{R}^{2n})$ with support in $[-T, T] \times \mathbb{S}^1$. Denote $X_T := [-T, T] \times \mathbb{S}^1$. Then in this case

$$\left|\left|\frac{\partial \xi}{\partial s}\right|\right|^2_{L^2(X)} + \left|\left|\frac{\partial \xi}{\partial t}\right|\right|^2_{L^2(X)} = \left|\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|\right|^2_{L^2(X)}. \tag{3.14}$$

To see this it suffices to assume $n = 1$, so $\xi = (f, g)$ and

$$\text{L.H.S.} = \text{R.H.S.} + 2 \int_{\mathbb{R}} \int_{\mathbb{S}^1} (\frac{\partial f}{\partial s}\frac{\partial g}{\partial t} - \frac{\partial g}{\partial s}\frac{\partial f}{\partial t}) dt\ ds.$$

Integration by parts gives

$$\int_{\mathbb{R}} \int_{\mathbb{S}^1} \frac{\partial g}{\partial s}\frac{\partial f}{\partial t} dt\ ds = - \int_{\mathbb{R}} \int_{\mathbb{S}^1} f \frac{\partial^2 g}{\partial s \partial t} dt\ ds.$$

On the other hand, by integrating in s first and applying by parts, since $f = 0$ for $s > T$,

$$\int_{\mathbb{S}^1} \int_{\mathbb{R}} \frac{\partial f}{\partial s}\frac{\partial g}{\partial t} ds\ dt = - \int_{\mathbb{R}} \int_{\mathbb{S}^1} f \frac{\partial^2 g}{\partial s \partial t} dt\ ds.$$

From

$$\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + 2S\xi\right|^2 \geq 0,$$

$$\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|^2 + \langle \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}, S\xi \rangle + |S\xi|^2 \geq \frac{1}{2}\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|^2 - |S\xi|^2.$$

So

$$\begin{aligned} \int_X |F\xi|^2 ds\ dt &= \int_{X_T} (\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S\xi\right|^2) ds\ dt \\ &\geq \int_{X_T} (\frac{1}{2}\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|^2 - |S\xi|^2) ds\ dt \\ &\geq \frac{1}{2} \int_{X_T} (\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|^2 - \tilde{c}\,|\xi|^2) ds\ dt \end{aligned}$$

where $\tilde{c} = \sup_{X_T} |S(s,t)|$. Hence

$$\begin{aligned} ||\xi||^2_{W^{1,2}(X)} &= \left|\left|\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t}\right|\right|^2_{L^2(X)} + ||\xi||^2_{L^2(X)} \quad \text{by (3.14)} \\ &\leq (\tilde{c} - 1)\, ||\xi||^2 + 2\, ||F\xi||^2 . \\ \Rightarrow ||\xi||_{W^{1,2}(X)} &\leq c(||\xi||_{L^2(X)} + ||F\xi||_{L^2(X)}). \end{aligned} \tag{3.15}$$

Now let $\beta \in C^\infty(\mathbb{R}, [0,1])$ be a cutoff function such that

$$\beta(s) = \begin{cases} 1 & \text{for } |s| \leq T-1, \\ 0 & \text{for } |s| \geq T. \end{cases}$$

Then

$$\begin{aligned} ||F(\beta\xi)||_{L^2(X)} &= ||\beta(\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S\xi) + \dot{\beta}\xi||_{L^2(X)} \\ &\leq ||F\xi||_{L^2(X)} + c||\xi||_{L^2(X_T)} \text{ for some } c > 0. \end{aligned} \tag{3.16}$$

Similarly,

$$||F((1-\beta)\xi)||_{L^2(X)} \leq ||F\xi||_{L^2(X)} + c||\xi||_{L^2(X_T)} \text{ for some } c > 0. \tag{3.17}$$

By Rellich compact embedding, the following composition is a compact operator,

$$K : W^{1,2}(X, \mathbb{R}^{2n}) \overset{\text{restriction}}{\longrightarrow} W^{1,2}(X_T, \mathbb{R}^{2n}) \overset{\text{compact}}{\hookrightarrow} L^2(X_T, \mathbb{R}^{2n})$$

$$K\xi := \xi|_{X_T}.$$

Therefore

$$
\begin{aligned}
&||\xi||_{W^{1,2}(X)}\\
\leq&||\beta\xi||_{W^{1,2}(X)}+||(1-\beta)\xi||_{W^{1,2}(X)}\\
\leq&c_1(||\beta\xi||_{L^2(X)}+||F(\beta\xi)||_{L^2(X)})+c_2||F((1-\beta)\xi)||_{L^2(X)} \text{ by step 2 and (3.15)}\\
\leq&c(||F\xi||_{L^2(X)}+||K\xi||_Z) \quad \text{by (3.16), (3.17)}
\end{aligned}
$$

Step 4.

We have shown F has finite dimensional kernel and has a closed range. So the cokernel of F satisfies the isomorphism

$$\mathrm{coker}F \cong R(F)^{\perp} \cong \ker F^*.$$

Observe that the adjoint operator F^* is given by

$$F^*\xi = -\frac{\partial \xi}{\partial s} + J_0\frac{\partial \xi}{\partial t} + S\xi.$$

The previous steps can be carried out for F^* and so $\ker F^*$ and hence $\mathrm{coker}F$ is also finite dimensional. This shows that F is Fredholm.

□

For the proof of the second part of proposition 3.13, we have to study the so called spectral flow of a family of operators. But first of all we have the following lemma which allows us to consider particular nice kind of Fredholm operator as in (3.6) given by some S which is easier to analyze. Suppose now we have another symmetric family of matrices $\tilde{S}(s,t)$ as before with

$$\lim_{s\to\pm\infty}\tilde{S}(s,t) = S^{\pm}(t).$$

Then by proposition 3.13, $\tilde{F} := \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + \tilde{S}$ is clearly also Fredholm. Denote the space of Fredholm operator from $W^{1,2}(X, \mathbb{R}^{2n})$ to $L^2(X, \mathbb{R}^{2n})$ by $\mathcal{F}$. Proposition 3.13 tells us that the operator of the form $F_S := \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S \in \mathcal{F}$. Define $\Sigma := \{F = F_S \in \mathcal{F}\}$ all the Fredholm operators of this form. We define $F_{\tilde{S}}$ to be equivalent to F_S if

$$\lim_{s \to \pm\infty} \tilde{S} = S^{\pm} = \lim_{s \to \pm\infty} S$$

and denote by Θ_S the equivalence class of F_S in Σ. Then

Lemma 3.15. *Θ_S is contractible within Σ as a subspace in $\mathcal{F}$.*

Proof. Take any $F_{S_0} \in \Sigma$ and let $\Theta = \Theta_{S_0}$. Define the homotopy to be

$$H : [0,1] \times \Theta \to \Theta \text{ by } (\tau, F_S) \mapsto F_{(1-\tau)S_0 + \tau S}.$$

For all $\tau \in [0,1]$, $F_{(1-\tau)S_0+\tau S} \in \Theta$ as $\lim_{s\to\pm\infty} (1-\tau)S_0 + \tau S = \lim_{s\to\pm\infty} S_0 = S_0^{\pm}$. It is also easy to check that it is continuous. $\square$

The significance of the above lemma is that the index map $\mu : \Sigma \to \mathbb{Z}$ is constant when restricted to the equivalence class Θ_S. In other words, $\mu(F_S)$ depends only on the endpoints $S^{\pm}$ of S. Now consider a continuous family of operator

$$A(s) : W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \to L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$$

for $s \in \mathbb{R}$ defined by

$$A(s)\xi(t) := J_0 \frac{d\xi}{dt} + S(s,t)\xi(t).$$

This is a family of symmetric operator defined on (a dense subset of) $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$.

These symmetric operators have a discrete spectrum consisting of real eigenvalues, each having finite multiplicity (see [33]). Also, the eigenvalues of $A(s)$ occur in continuous families $\lambda_j(s)$ for $j \in \mathbb{Z}$ counted with multiplicity. The limit operator $A^{\pm} = \lim_{s \to \pm\infty} A(s)$ is invertible by (3.8). The Fredholm index of (3.6) is then given by the spectral flow of A (see [3], [28]), which roughly speaking measures the algebraic increase of eigenvalues of $A(s)$ flowing from negative to positive, as s goes form $-\infty$ to ∞. More precisely,

$$\operatorname{index} F = \#\{j : \lambda_j(-\infty) < 0 < \lambda_j(\infty)\} - \#\{j : \lambda_j(-\infty) > 0 > \lambda_j(\infty)\}.$$

Now we want to prove that the spectral flow agrees with $\mu(\Psi^-) - \mu(\Psi^+)$. This would prove the remaining part of proposition 3.13.

Proof of Proposition 3.13 (2). In each homotopy class of $\mathcal{LP}^*$, there exists a path of the form $\Psi(t) \in Sp(2n, \mathbb{R}) = \exp(J_0 St) \in Sp(2n, \mathbb{R})$ with $\Psi(1) = W^{\pm}$, where S is a constant real symmetric matrix. More precisely, recall that by proposition 3.7, each homotopy class in $\mathcal{LP}^*$ is characterized by the Maslov index $\mu(\Psi) = k$.

For odd $n-k$, by decomposing $\mathbb{R}^{2n}$ as $(\mathbb{R}^2)^n$, choose

$$S = \begin{pmatrix} 0 & \log 2 \\ \log 2 & 0 \end{pmatrix} \oplus \bigoplus_{j=1}^{n-1} \begin{pmatrix} m_j\pi & 0 \\ 0 & m_j\pi \end{pmatrix} = \bigoplus_{j=1}^{n} S_j \tag{3.18}$$

where $m_1 = n - k - 2$ and $m_j = -1$ otherwise.

For even $n-k$, choose

$$S = \begin{pmatrix} -\pi & 0 \\ 0 & -\pi \end{pmatrix} \oplus \bigoplus_{j=1}^{n-1} \begin{pmatrix} m_j\pi & 0 \\ 0 & m_j\pi \end{pmatrix} = \bigoplus_{j=1}^{n} S_j \tag{3.19}$$

where $m_1 = n-k-1$ and $m_j = -1$ otherwise.

We will prove that for these S, the induced $\Psi(t) = \exp(J_0 St)$ has $\mu(\Psi) = k$ and $\Psi(1) = W^{\pm}$, in particular $\Psi \in \mathcal{LP}^*$. Thus each path in $\mathcal{LP}^*$ is homotopic to exactly one of these Ψ. Indeed if $\Psi(1) = W^{\pm}$, then by definition 3.6 of μ and the product and determinant property in proposition 3.2, $\mu(\Psi) = \sum_{j=1}^{n} \mu(\Psi_j)$, where $\Psi_j(t) := \exp(J_0 S_j t)$. By lemma 3.15, $\operatorname{index} F$ depends only on the endpoints $S^{\pm}$, which in particular can be chosen in the form of (3.18) or (3.19). So now let

$$S(s) = \beta(s)S^+ + (1-\beta(s))S^-$$

where $S^{\pm}$ is of the above form and $\beta \in C^\infty(\mathbb{R}, [0,1])$ is an non-decreasing smooth function such that

$$\beta(s) = \begin{cases} 1 & \text{if } s \geq 1 \\ 0 & \text{if } s \leq -1 \end{cases}$$

We now study the spectral flow for $A(s)$ given by this $S(s)$. It also suffices to decompose the matrix $\Psi(s,t) = \exp(J_0 S(s) t) \in Sp(2n, \mathbb{R})$ into 2×2 blocks. So we can assume $n = 1$. There are three cases.

Case (i)

$$S^- = \begin{pmatrix} -k^-\pi & 0 \\ 0 & -k^-\pi \end{pmatrix}, \quad S^+ = \begin{pmatrix} -k^+\pi & 0 \\ 0 & -k^+\pi \end{pmatrix} \quad \text{where } k^\pm \text{ are odd.}$$

Then

$$\Psi^\pm(t) = \exp(J_0 S^\pm t) = \begin{pmatrix} \cos(k^\pm\pi t) & -\sin(k^\pm\pi t) \\ \sin(k^\pm\pi t) & \cos(k^\pm\pi t) \end{pmatrix} \in U(1)$$

By the determinant property in proposition 3.2, $\rho(\Psi^\pm) = e^{ik^\pm\pi t}$, so $\Psi(1) = -I = W^+$ and $\mu(\Psi^\pm) = k^\pm$. Now consider

$$\begin{aligned}
\lambda \in \sigma(A(s)) \quad &\Leftrightarrow \quad J_0\frac{d\xi}{dt} + S(s)\xi = \lambda\xi \text{ for some } \xi \neq 0 \\
&\Leftrightarrow \quad \frac{d\xi}{dt} = J_0(S(s) - \lambda I)\xi \\
&\Leftrightarrow \quad \xi(t) = \exp(J_0(S(s) - \lambda I)t)\xi(0) \quad \text{ for some } \xi(0) \neq 0 \\
&\Leftrightarrow \quad 1 \in \sigma(\exp(J_0(S(s) - \lambda I))) \quad \text{ as } \xi(1) = \xi(0) \neq 0.
\end{aligned}$$

$$S(s) = \begin{pmatrix} -\omega(s) & 0 \\ 0 & -\omega(s) \end{pmatrix} \qquad \text{where } \omega(s) = (\beta(s)k^+ + (1 - \beta(s))k^-)\pi.$$

So

$$\exp(J_0(S(s) - \lambda I)t) = \begin{pmatrix} \cos(\omega(s) + \lambda)t & -\sin(\omega(s) + \lambda)t \\ \sin(\omega(s) + \lambda)t & \cos(\omega(s) + \lambda)t \end{pmatrix}.$$

This implies $1 \in \sigma(\exp(J_0(S(s) - \lambda I))) \Leftrightarrow \omega(s) + \lambda \in 2\pi\mathbb{Z}$. Therefore the family of eigenvalues of $A(s)$ are exactly

$$\lambda_j(s) = -\omega(s) + 2\pi j \quad \text{where } j \in \mathbb{Z}$$

and each $\lambda_j(s)$ occurs in multiplicity 2. As $\omega(s)$ varies monotonically from $k^-\pi$ to $k^+\pi$ and $k^\pm$ are odd, there are exactly $\frac{|k^- - k^+|}{2}$ values (not counting multiplicities) of j such that $0 \in \sigma(A(s))$ for some s, i.e. these eigenvalues of $A(s)$ will cross the zero. (For example if $k^- = -3$, $k^+ = 1$, then all these j are $j = 0, 1$). If $k^- > k^+$, then $\lambda_j(s)$ increase when s goes from $-\infty$ to ∞, otherwise if $k^- < k^+$ they decrease. Thus the spectral flow of $A(s)$ is $k^- - k^+$.
Similar argument shows that for case (ii) where

$$S^- = \begin{pmatrix} 0 & \log 2 \\ \log 2 & 0 \end{pmatrix} \quad \text{and} \quad S^+ = \begin{pmatrix} -\pi & 0 \\ 0 & -\pi \end{pmatrix}, \text{ we have}$$

$$\mu(\Psi^-) = 0 \quad \text{and} \quad \mu(\Psi^+) = 1.$$

There is only one eigenvalue $\lambda(s) = (1-\beta(s))\log 2 - \beta(s)\pi$ of $A(s)$ crossing zero (with constant eigenfuction $\xi(t) \equiv (1,1)$). Thus the spectral flow is -1 which agrees with $\mu(\Psi^-) - \mu(\Psi^+)$.
The remaining case (iii) is the same as case (ii) except the roles of S^+ and S^- are switched. Thus by reversing time s we get the same conclusion. $\square$

Consider the more general operator $F : W^{1,2}(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}), \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}), \mathbb{R}^{2n})$ by

$$F\xi = \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + (A+S)\xi \tag{3.20}$$

where S is symmetric as before and $A = A(s,t) \in M(2n, \mathbb{R})$ is continuous matrix valued and is skew symmetric for all s, t. Suppose

$$\lim_{s\to\pm\infty} A(s,t) = 0$$

uniformly in t. Then this operator F is a compact perturbation of (3.6) and so is also Fredholm of the same index.

Now recall that for a pair of non-degenerate solutions $x^{\pm} \in P(H)$ of (2.1) and $u \in C^{\infty}(\mathbb{R} \times \mathbb{S}^1, M)$ such that

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t), \tag{3.21}$$

$$\lim_{s\to\pm\infty} \frac{\partial u}{\partial t} = \dot{x}^{\pm}(t) \quad \text{and} \quad \lim_{s\to\pm\infty} \frac{\partial u}{\partial s} = 0 \tag{3.22}$$

where all limits are uniform in t, the linear operator $F(u) : W^{1,2}(u^*TM) \to L^2(u^*TM)$ is defined by

$$F(u)\xi := \nabla_s \xi + J(u)\nabla_t \xi + (\nabla_\xi J(u))\frac{\partial u}{\partial t} + \nabla_\xi \nabla H_t(u). \tag{3.23}$$

Theorem 3.16. *Suppose $u : \mathbb{R} \times \mathbb{S}^1 \to M$ satisfies* (3.21), (3.22) *for a pair of non-degenerate solutions $x^{\pm} \in P(H)$ of* (2.1). *Then $F(u)$ is Fredholm and its index is given by*

$$\operatorname{index} F(u) = \mu(x^-) - \mu(x^+).$$

Proof. The key is to use a compact perturbation if necessary, and using local coordinates, alter the operator to the form in (3.6). Then we can apply proposition 3.13 to get the index as the difference of the respective Maslov indices.

By altering u if necessary, we can assume that

$$u(s,t) = \begin{cases} x^-(t) & \text{if } x \leq -1 \\ x^+(t) & \text{if } x \geq 1. \end{cases}$$

This would not change the Fredholm index. By proposition 3.8, there is a symplectic orthogonal trivialization

$$\Phi = \Phi(s,t) : \mathbb{R}^{2n} \to T_{u(s,t)}M$$

In this local coordinates, the operator $F(u)$ is represented by

$$F := \Phi^{-1}F(u)\Phi : W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$$

$$F = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + (S + A),$$

where S and A denotes the symmetric and anti-symmetric part of the matrix given by $\langle Z_i, F(u)Z_j \rangle$ and $Z_i = Z_i(s,t) \in T_{u(s,t)}M$ are the orthonormal frames given by the trivialization Φ. (A side remark: the asymptotic operators $J_0 \frac{\partial}{\partial t} + S(\pm\infty)$ is the Hessian of A_H at $x^\pm$ in the trivialization $\Phi(\pm\infty)$, so F can be regarded as one-parameter family of operators which is asymptotically symmetric.) By direct calculations,

$$A_{ij} = \langle Z_i, \nabla_s Z_j \rangle = -A_{ji} \quad \text{and}$$

$$S_{ij} = \langle Z_i, (\nabla_{Z_j} J)\frac{\partial u}{\partial t} + \nabla_{Z_j} \nabla H_t + J \nabla_t Z_j \rangle = S_{ji}. \tag{3.24}$$

As $Z_i(s,t) \equiv Z_i(\pm 1, t)$ for $\pm s \geq 1$, $A(s,t) = 0$ for $|s| \geq 1$. So by a compact perturbation we can assume that $A(s,t) = 0$ for all s, t.

So the operator becomes

$$F = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S.$$

Let $S^{\pm} = S(\pm, t)$, $\Phi^{\pm}(t) = \Phi(\pm 1, t)$ and $\Psi^{\pm}(t) \in Sp(2n, \mathbb{R})$ be given by (3.5). It remains to show $\Psi^{\pm}$ satisfies (3.7). Clearly we only have to show it for Ψ^{+}, so to simplify the notations, denote $S = S^{+}$, $\Phi_t = \Phi(t) = \Phi^{+}(t)$, $x_0 = x^{+}(0)$, $\Psi_t = \Psi^{+}(t)$, $x(t) = x^{+}(t)$ and $d\psi_t = d\psi_t(x^{+}(0))$. Let $v \in \mathbb{R}^{2n}$. By definition,

$$\Phi_t \Psi_t v = d\psi_t \Phi_0 v. \tag{3.25}$$

Applying covariant derivative with respect to t on R.H.S.,

$$\begin{aligned} \nabla_t(d\psi_t \Phi_0 v) &= \nabla_t \nabla_{s=0}(\psi_t \circ \exp_{x_0}(s\Phi_0 v)) \\ &= \nabla_{s=0} \nabla_t(\psi_t \circ \exp_{x_0}(s\Phi_0 v)) \\ &= \nabla_{d\psi_t \Phi_0 v} X(x(t)) \\ &= \nabla_{\Phi_t \Psi_t v} X. \end{aligned}$$

So differentiating (3.25) gives

$$\Phi_t \dot{\Psi}_t v + (\nabla_t \Phi_t) \Psi_t v = \nabla_{\Phi_t \Psi_t v} X.$$

Consider

$$\begin{aligned}
\Phi_t J_0 \dot{\Psi}_t v &= J\Phi_t \dot{\Psi}_t v \quad (\text{as } \Phi_t J_0 = J\Phi_t) \\
&= J(\nabla_{\Phi_t \Psi_t v} X - (\nabla_t \Phi_t)\Psi_t v) \\
&= J(\nabla_{\Phi_t \Psi_t v}(J\nabla H) - (\nabla_t \Phi_t)\Psi_t v) \\
&= J(\nabla_{\Phi_t \Psi_t v} J)\nabla H - \nabla_{\Phi_t \Psi_t v}(\nabla H) - J(\nabla_t \Phi_t)\Psi_t v \\
&= -\nabla_{\Phi_t \Psi_t v} J(J\nabla H) - \nabla_{\Phi_t \Psi_t v}(\nabla H) - J(\nabla_t \Phi_t)\Psi_t v \\
&= -(\nabla_{\Phi_t \Psi_t v} J)(X) - \nabla_{\Phi_t \Psi_t v}(\nabla H) - J(\nabla_t \Phi_t)\Psi_t v \\
&= -\Phi_t S(t)\Psi_t v \quad \text{by (3.24)}.
\end{aligned}$$

Therefore

$$\dot{\Psi}^{\pm}(t) = J_0 S^{\pm} \Psi^{\pm}(t).$$

By proposition 3.13,

$$\operatorname{index} F(u) = \operatorname{index} F = \mu(\Psi^-) - \mu(\Psi^+) = \mu(x^-) - \mu(x^+).$$

$\square$

Chapter 4

Floer Homology

In this chapter, we will look more closely at the Floer homology groups of Hamiltonian function for symplectic manifolds. We will provide some details of the proofs of the invariance of Floer homology and the isomorphism between Floer homology and singular homology of M.

4.1 Transversality

Recall that $\mathcal{M}$ is the set of bounded solution to (2.7) and (H, J) is called a **regular pair** if

1. All contractible $x \in P(H)$ are non-degenerate, and
2. If $x^{\pm} \in P(H)$ are contractible and $u \in \mathcal{M}(x^{-}, x^{+})$, then $F(u)$ is surjective.

Proposition 4.1. *There is a dense subset of smooth almost complex structure $\mathcal{J}_{\text{reg}} \subset C^{\infty}(\text{End}(\text{TM}))$ such that for all $J \in \mathcal{J}_{\text{reg}}$ and $u \in$*

$\mathcal{M}$, $F(u)$ is onto, i.e. (H, J) is regular.

The proof can be found in [9] and [11]. It uses a result of Smale [35], which generalizes the Sard's theorem into the infinite dimensional case.
Using this result, by choosing (H, J) to be regular, and using the implicit function theorem, it follows that there is a neighborhood of u in $\mathcal{M}(x, y)$ which is diffeomorphic to a neighborhood of zero in $\ker F(u)$. Furthermore, by theorem 3.16, we have

$$\begin{aligned} \dim \mathcal{M}(x, y) &= \operatorname{index} F(u) \\ &= \mu(x) - \mu(y). \end{aligned} \tag{4.1}$$

4.2 Compactness and Gluing

In this section we fix a smooth Hamiltonian function $H = H(t, x)$ and $J \in \mathcal{J}$ and denote $\mathcal{M}(x, y; H, J)$ simply by $\mathcal{M}(x, y)$. Define the space of bounded solution to (2.7)

$$\mathcal{M} := \{u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M) : u \text{ solves (2.7) and is contractible, } E(u) < \infty\}.$$

Then theorem 2.14 can be restated as

Theorem 4.2. $\mathcal{M} = \displaystyle\bigcup_{x,y \in P(H)} \mathcal{M}(x, y)$.

$\mathcal{M}$ also has the following compactness property.

Theorem 4.3 (Gromov compactness). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(x,y)$. Then there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and sequences of time $(s_k^i)_{k\in\mathbb{N}}$, $i=0,\cdots,m$ such that*

$$\lim_{k\to\infty} u_{n_k}(s+s_k^i,t) = u^i(s,t) \tag{4.2}$$

where $u^i \in \mathcal{M}(x^i,x^{i+1})$, $x^i \in P(H)$, $x^0 = x, x^{m+1} = y$. The convergence is uniform in all derivatives on compact subset (i.e. in C^∞_{loc} topology). If (H,J) is a regular pair, then

$$\mu(x^i) > \mu(x^{i+1}) \quad \textit{for } i = 0,\cdots,m.$$

Remark 4.4. *The convergence as in theorem 4.3 (4.2), $m \geq 1$, is called a **geometric convergence** (or **weak convergence**) **towards a broken trajectory of order** m.*

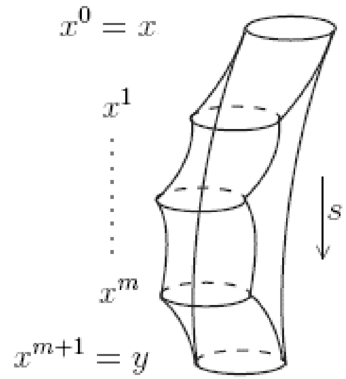


Figure 4.1: Convergence towards a broken trajectory.

The geometric convergence for a sequence of unparametrized trajectory $\hat{u}_n = [u_n] \in \hat{\mathcal{M}}(x,y)$ is defined analogously, and is denoted as

$$\hat{u}_n \rightharpoonup (\hat{u}^0,\cdots,\hat{u}^m).$$

If $\mu(x) - \mu(y) = 2$ and the order $m = 1$ then we say it ***converges to a simply broken trajectory****.*

We emphasize here that in theorem 4.3, our assumption (2.3) that ω vanishes on $\pi_2(M)$ is essential, otherwise another kind of limiting behavior, called bubbling, would occur. The "$\supset$" part of theorem 4.2 has been proved in theorem 2.14. For "$\subset$", we need the following two lemmas.
Denote the open disk of radius r by $B_r := \{z \in \mathbb{C} : |z| < r\}$.

Lemma 4.5. *There exists a constant $\varepsilon = \varepsilon(M, \omega, H, J) > 0$ such that for all solution $u \in C^\infty(B_r, M)$ of (2.7) with*

$$\int_{B_r} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds \leq \varepsilon,$$

then

$$\left|\frac{\partial u}{\partial s}(0)\right| \leq 1 + \frac{8}{\pi r^2} \int_{B_r} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds.$$

By lemma 4.5, for $u \in \mathcal{M}$, since $\left|\frac{\partial u}{\partial t}(s,t)\right| \leq \left|\frac{\partial u}{\partial s}(s,t)\right| + |\nabla H(t,u)|$ and M is compact,

$$||\nabla u||_{L^\infty(\mathbb{R}\times\mathbb{S}^1)} := \sup_{s\in\mathbb{R}, t\in\mathbb{S}^1} \{\max(\frac{\partial u}{\partial s}(s,t), \frac{\partial u}{\partial t}(s,t))\} < \infty. \qquad (4.3)$$

For $u : \mathbb{R}\times\mathbb{S}^1 \cong \mathbb{C}/i\mathbb{Z} \to M$, we can also regard u as $u = u(s+it) \in C^\infty(\mathbb{C}, M)$. Of course it does not really matter if we are concerning about $||\cdot||_{L^\infty}$.

Lemma 4.6. *Let $\Omega \subset \mathbb{C}$ be an open domain. Then every sequence of solutions $u_n \in C^\infty(\Omega, M)$ of (2.7) such that*

$$\sup_{n \in \mathbb{N}} ||\nabla u_n||_{L^\infty(\Omega)} < \infty$$

has a subsequence converging in C^∞_{loc} topology.

Note that such (sub-) sequence u_n converges to some $u \in \mathcal{M}$. For if $p = u(s_0, t_0) \in M$, by choosing a chart around p to some relatively compact domain $\Omega \subset \mathbb{R}^{2n}$ and by restricting the domain of u_n, we can assume that $u_n \in C^\infty(\mathbb{R} \times \mathbb{S}^1, \Omega)$ such that

$$\frac{\partial u_n}{\partial s} + J(u_n)\frac{\partial u_n}{\partial t} + \nabla H_t(u_n) = 0.$$

Then for $\varepsilon > 0$,

$$\begin{aligned} & \left|\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H_t(u)\right| \\ = \; & \left|(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H_t(u)) - (\frac{\partial u_n}{\partial s} + J(u_n)\frac{\partial u_n}{\partial t} + \nabla H_t(u_n))\right| \\ < \; & \epsilon \end{aligned}$$

for sufficiently large n by the convergence of u_n and uniform continuity of H and J. Since $\varepsilon > 0$ is arbitrary we conclude that u satisfies (2.7).

The proofs of lemma 4.5 and lemma 4.6 can be found in [30].

Proof of theorem 4.2. We have to prove that $\mathcal{M} = \bigcup_{x,y \in P(H)} \mathcal{M}(x, y)$. Suppose not, then there exists $u \in \mathcal{M}$, $\varepsilon > 0$ and a sequence

$(s_n, t_n) \in \mathbb{R} \times \mathbb{S}^1$ such that $|s_n| \to \infty$ and for all $x \in P(H)$, $n \in \mathbb{N}$,

$$d(u(s_n, t_n), x(t_n)) \geq \varepsilon$$

where d denotes the distance in M induced by its Riemannian metric. Define $u_n(s,t) := u(s + s_n, t)$. Then by lemma 4.6 there exists a subsequence, which we still denote by u_n for convenience, converging to some $v \in \mathcal{M}$. Assume also that t_n converges to t_0. Then

$$d(v(0, t_0), x(t_0)) \geq \varepsilon \tag{4.4}$$

for any $x \in P(H)$. Since $|s_n| \to \infty$, as $\int_{\mathbb{R}} \int_{\mathbb{S}^1} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds < \infty$,

$$\begin{aligned} \int_{-T}^{T} \int_{\mathbb{S}^1} \left|\frac{\partial v}{\partial s}\right|^2 dt\, ds &= \lim_{n\to\infty} \int_{-T}^{T} \int_{\mathbb{S}^1} \left|\frac{\partial u_n}{\partial s}\right|^2 dt\, ds \\ &= \lim_{n\to\infty} \int_{-T}^{T} \int_{\mathbb{S}^1} \left|\frac{\partial u}{\partial s}(s + s_n, t)\right|^2 dt\, ds = 0 \end{aligned}$$

for all $T > 0$. So $\frac{\partial v}{\partial s} = 0$ and hence v is independent of s, but then $v(s,t) = x(t)$ for some $x \in P(H)$, this contradicts (4.4). □

Proof of theorem 4.3. This proof is from [30].

Step 1. We first claim that

$$\sup_{u \in \mathcal{M}} ||\nabla u||_{L^\infty} < \infty$$

We prove by contradiction. Suppose the contrary, then there exists a sequence $u_n \in \mathcal{M}$ such that

$$c_n := ||\nabla u_n||_{L^\infty} \to \infty.$$

We assume the domain of u_n is $\mathbb{C}$ for convenience. So there exists $z_n = s_n + it_n$ such that

$$\max\left\{\left|\frac{\partial u_n}{\partial s}(z_n)\right|, \left|\frac{\partial u_n}{\partial t}(z_n)\right|\right\} \geq \frac{c_n}{2}.$$

Define $v_n(z) := u_n(z_n + \frac{1}{c_n}z)$ and denote $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. Then

$$|\nabla v_n(0)| \geq \frac{1}{2}, \quad ||\nabla v_n||_{L^\infty} \leq 1, \tag{4.5}$$

$$\frac{\partial v_n}{\partial s} + J(v_n)\frac{\partial v_n}{\partial t} + \frac{1}{c_n}\nabla H(v_n, t_n + \frac{t}{c_n}) = 0, \quad \text{and} \tag{4.6}$$

$$\begin{aligned} \int_{B_{c_n}(0)} \left|\frac{\partial v_n}{\partial s}\right|^2 &= \int_{B_1(z_n)} \left|\frac{\partial u_n}{\partial s}\right|^2 \\ &\leq 2E(u_n) \quad \text{(by definition 2.13)} \\ &\leq 2 \max_{x,y \in P(H)} |A(x) - A(y)| \quad \text{(by (2.10))}. \end{aligned} \tag{4.7}$$

From (4.5) and by lemma 4.6, v_n converges to some $v \in C^\infty(\mathbb{C}, M)$ such that

$$\nabla v(0) \neq 0, \tag{4.8}$$

$$\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0, \quad \text{and} \tag{4.9}$$

$$0 < \int_{\mathbb{C}} \left|\frac{\partial v}{\partial s}\right|^2 < \infty. \tag{4.10}$$

Define $\gamma_r : \mathbb{S}^1 \to M$ by $\gamma_r(\theta) := v(re^{i2\pi\theta})$, observe that

$$|\dot{\gamma}_r(\theta)| = 2\pi r \left|\frac{\partial v}{\partial s}(re^{i2\pi\theta})\right|.$$

So

$$\int_0^\infty \frac{1}{2\pi r}||\dot{\gamma}_r||^2_{L^2(\mathbb{S}^1)}dr = \int_0^\infty \frac{1}{2\pi r}\int_0^1 |\dot{\gamma}_r(\theta)|^2 d\theta\, dr = \int_{\mathbb{C}} \left|\frac{\partial v}{\partial s}\right|^2 < \infty.$$

As the length of γ_r, $l(\gamma_r) = \int_0^1 |\dot{\gamma}_r| \cdot 1\, d\theta \leq (\int_0^1 |\dot{\gamma}_r|^2 d\theta)^{\frac{1}{2}} = ||\dot{\gamma}_r||_{L^2}$ by the Cauchy-Schwarz inequality, by choosing a sufficiently large $R > 0$, $l(\gamma_R)$ can be arbitrary small. Choose a symplectic chart $h : U \to \mathbb{R}^{2n}$ of M such that $\gamma_r(\mathbb{S}^1) \subset U$, $h(U)$ is a bounded convex domain and $h(\gamma_r(1)) = 0$.

Define $w : \mathbb{S}^2 = \mathbb{C} \cup \{\infty\} \to M$ by

$$w(re^{i2\pi\theta}) := \begin{cases} v(re^{i2\pi\theta}) & r \leq R \\ h^{-1}(\frac{R}{r} h \circ v(Re^{i2\pi\theta})) & r \geq R. \end{cases}$$

Let $\tilde{w}(\rho, \theta) := \rho R \gamma_R(\theta)$, $\varepsilon > 0$ and consider

$$\begin{aligned} \int_{\mathbb{S}^2 - B_R} w^*\omega &= -\int_{B_{\frac{1}{R}}} \tilde{w}^*\omega_0 \\ &= \int_0^{1/R} \int_0^1 \omega_0(R\gamma_R, \rho R \dot{\gamma}_R) d\rho\, d\theta \\ &= R^2 \int_0^{1/R} \int_0^1 \rho g(\gamma_R, -J\dot{\gamma}_R) d\rho\, d\theta \\ &= R^2 \int_0^{1/R} \int_0^1 \rho |\gamma_R||\dot{\gamma}_R| d\rho\, d\theta \\ &\leq c \cdot l(\gamma_R) < \varepsilon \end{aligned}$$

for sufficiently large R where the constant $c > 0$ depends on $h(U)$ only. Therefore

$$\int_{\mathbb{S}^2} w^*\omega \geq \int_{B_R} v^*\omega - \varepsilon(R) = \int_{B_R} \left| \frac{\partial v}{\partial s} \right|^2 - \varepsilon(R) > 0$$

for sufficiently large R, the last equation follows from $\omega(\frac{\partial v}{\partial s}, \frac{\partial v}{\partial t}) = \omega(\frac{\partial v}{\partial s}, J\frac{\partial v}{\partial s}) = |\frac{\partial v}{\partial s}|^2$. This contradicts our assumption that $\omega(\pi_2(M)) =$

0. Our claim is proved.

Step 2. Since $P(H)$ are isolated, there exists $\varepsilon > 0$ such that $\sup_{t\in\mathbb{S}^1} d(x(t), y(t)) > 2\varepsilon$ for all $x, y \in P(H)$. Given a sequence u_n in $\mathcal{M}(x, y)$, clearly we can assume $x \neq y$, for otherwise $x = y$ implies $E(u_n) = A(x) - A(y) = 0$ and so $u_n \equiv x$ for all n by remark 2.16 and we have nothing to prove. Define

$$s_n^1 := \inf\{s \in \mathbb{R} : d(u_n(s,t), x(t)) > \varepsilon \text{ for some } t \in \mathbb{S}^1\}.$$

The sets which we are taking infimum at are all non-empty as $\lim_{s\to\infty} u_n(s,t) = y(t)$ and $s_n^1 \neq -\infty$ by a similar reason, as $\lim_{s\to-\infty} u_n(s,t) = x(t)$. By step 1 and lemma 4.6, there is a subsequence (which for convenience taken to be itself), such that $u_n^1(s,t) := u_n(s + s_n^1, t)$ converges to some $u^1 \in \mathcal{M}$. By theorem 4.2 (1), $u^1 \in \mathcal{M}(x^0, x^1)$ for some $x^0, x^1 \in P(H)$. Since $d(u^1(s,t), x(t)) \leq \varepsilon$ for all $s \leq 0$, $x^0 = x$. Also $x^1 \neq x$, for if otherwise, $u^1 \equiv x(t)$ again by remark 2.16. But by the definition of u_n^1 there exists $t_0 \in \mathbb{S}^1$ such that $d(u^1(0,t_0), x(t_0)) \geq \varepsilon$, a contradiction. If $x^1 = y$, then we are done. Otherwise we prove by induction, i.e. we claim that if we have $u^i \in \mathcal{M}(x^{i-1}, x^i)$, $\lim_{n\to\infty} u_n(s + s_n^i, t) = u^i(s,t)$ for $i = 0, \cdots, k$ with $x^k \neq y$, then there exists $u^{k+1} \in \mathcal{M}(x^k, x^{k+1})$ and a sequence s_n^{k+1} such that $\lim_{n\to\infty} u_n(s + s_n^{k+1}, t) = u^{k+1}(s,t)$ for some $x^{k+1} \in P(H)$, $x^{k+1} \neq x^k$.

Since $u^k \in \mathcal{M}(x^{k-1}, x^k)$, there exists s_0 such that if $s \geq s_0$, $d(u^k(s,t), x^k(t)) < \varepsilon$ for all $t \in \mathbb{S}^1$. Then for sufficiently large n, for all $t \in \mathbb{S}^1$,

$d(u_n^k(s_0,t),x^k(t)) = d(u_n(s_0+s_n^k,t),x^k(t)) < \varepsilon$. So by passing into subsequence, define

$$s_n^{k+1} := \sup\{s : s \geq s_n^k + s_0, d(u_n(s,t),x^k(t)) < \varepsilon \text{ for all } t \in \mathbb{S}^1\}.$$

Then without loss of generality the sequence $u^{k+1}(s,t) := u_n(s+s_n^{k+1})$ converges to $u^{k+1} \in \mathcal{M}$ by step 1 and lemma 4.6. We claim that $u^{k+1} \in \mathcal{M}(x^k,x^{k+1})$ with $x^{k+1} \neq x^k$.

The sequence $s_n^{k+1} - s_n^k \to \infty$ for otherwise $[s_0, s_n^{k+1} - s_n^k]$ is contained in a compact interval $[s_0,s_1]$, then for each t, $u_n^{k+1}(0,t) = u_n^k(s_n^{k+1} - s_n^k, t)$ will converge to a point in $u^k([s_0,s_1] \times \mathbb{S}^1)$, so $d(u^{k+1}(0,t),x^k(t)) < \varepsilon$. But by our construction there exists $t_0 \in \mathbb{S}^1$ such that $d(u^{k+1}(0,t_0),x^k(t_0)) \geq \varepsilon$, a contradiction. So $s_n^k - s_n^{k+1} + s_0 \to -\infty$ and $\lim_{n\to\infty} u_n^{k+1}(s_n^k - s_n^{k+1} + s_0, t) = \lim_{n\to\infty} u_n^k(s_0,t) = u^k(s_0,t)$ with $d(u^k(s_0,t),x^k(t)) < \varepsilon$. It follows that $\lim_{s\to-\infty} u^{k+1}(s,t) = x^k(t)$. $x^{k+1} \neq x^k$ by the same reason as before.

By remark 2.16, if $x^k \neq x^{k+1}$ and $u^k \in \mathcal{M}(x^k,x^{k+1})$, $A(x^k) > A(x^{k+1})$, i.e. the action decreases. Since $P(H)$ is finite, this process must terminate to arrive at $x^{m+1} = y$, $m < \infty$.

Finally if (H,J) is regular, as $x^i \neq x^{i+1}$ for all i, the moduli space $\mathcal{M}(x^i,x^{i+1}) \ni u^i$ is at least one-dimensional and it follows from the dimension formula (4.1) $\dim \mathcal{M}(x^i,x^{i+1}) = \mu(x^i) - \mu(x^{i+1})$ that $\mu(x^i) > \mu(x^{i+1})$. □

If u_n does not converge to broken trajectories (of order $m \geq 1$), then its convergence is stronger:

Proposition 4.7. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(x,y)$ converging to $u \in \mathcal{M}(x,y)$ in C^∞_{loc} sense,*

$$\lim_{n\to\infty} u_n(s,t) = u(s,t).$$

Then it also converges in C^∞ sense, i.e. it converges uniformly in all derivatives.

The idea is that for fixed ends x and y, the non-degeneracy of x and y implies uniform exponential convergence of the ends of the trajectories. Away from the two ends, the uniform convergence is ensured by theorem 4.3.

Proposition 4.8. *Let (H,J) be a regular pair and $x,y \in P(H)$ with $\mu(x)-\mu(y)=1$, then the 0-dimensional manifold $\hat{\mathcal{M}}(x,y)$ is compact. i.e. it consists of finite number of points. In other words the set of trajectories between x and y is finite (modulo shifting).*

Proof. Let $u_n \in \mathcal{M}(x,y)$. Then by theorem 4.3, without loss of generality we can assume that

$$\lim_{n\to\infty} u_n(s,t) = u(s,t)$$

in C^∞_{loc} sense where $u(s,t) \in \mathcal{M}(x,y)$ (since $\mu(x)-\mu(y) = 1$, it cannot converge to a broken trajectory.) By proposition 4.7, u_n converges uniformly to u and thus $[u_n] \to [u]$. Therefore $\hat{\mathcal{M}}(x,y)$ is compact. $\square$

We have to analyze the one-dimensional moduli space of unparametrized trajectories of relative index 2 in order to prove that the boundary operator really defines Floer homology. We need a gluing construction due to Floer.

The gluing construction can be considered as a converse of Gromov's compactness (theorem 4.3), which states that any sequence $u_n \in \mathcal{M}(x, y)$ not converging in $\mathcal{M}(x, y)$ must converge (up to a subsequence) to a broken trajectory of some order m. The gluing construction tells us that we can reverse this process, i.e. we can "glue" a broken trajectory of order m $(u^0, \cdots, u^m) \in \mathcal{M}(x^0, x^1) \times \cdots \times \mathcal{M}(x^m, x^{m+1})$ together to get a trajectory in $\mathcal{M}(x^0, x^{m+1})$, up to m gluing parameters (which in some sense measures how close the resulting trajectory with each u^i is). For simplicity we will only give the statement for gluing a simply broken trajectory, which is sufficient in our treatment.

Proposition 4.9 (Floer's Gluing)**.** *(Unparametrized version)*
Let (H, J) be regular and $K \subset \hat{\mathcal{M}}(x, y) \times \hat{\mathcal{M}}(y, z)$ be a compact subset. Then there exists a constant $\rho_0 = \rho_0(K)$ and a gluing map

$$\begin{aligned} \# : K \times [\rho_0, \infty) &\rightarrow \hat{\mathcal{M}}(x, z) \\ (\hat{u}, \hat{v}, \rho) &\mapsto \hat{u}\#_\rho \hat{v} \end{aligned}$$

such that

1. *$\#$ is an embedding;*

2. *$\hat{u}\#_\rho\hat{v}$ converges to the broken trajectory $(\hat{u},\hat{v})$ geometrically as $\rho\to\infty$ (see remark 4.4),*

$$\hat{u}\#_\rho\hat{v} \rightharpoonup (\hat{u},\hat{v})$$

3. *for a sequence $(\hat{u}_n)_{n\in\mathbb{N}}$ of unparametrized trajectories in $\hat{\mathcal{M}}(x,z)$ converging geometrically to the simply broken trajectory $(\hat{u},\hat{v})$, then for sufficiently large n, $\hat{u}_n$ lies within the range of $\#$.*

The details can be found in [11] and also [8]. The idea is that we can first "pre-glue" u and v at y to get an approximate solution $u\hat{\#}_\rho v$ of (2.7) such that $||\overline{\partial}(u\hat{\#}_\rho v)||_{L^p} \leq e^{-c\rho}$ for large enough ρ where $c = c(K) > 0$. Explicitly this can be done by

$$u\hat{\#}_\rho v(s,t) := \begin{cases} u(s+\rho,t) & ,s\leq -1 \\ \exp_{y(t)}(\beta(-s)\xi(s+\rho,t)+\beta(s)\zeta(s-\rho,t)) & ,s\in[-1,1] \\ u(s-\rho,t) & ,s\geq 1. \end{cases}$$

where ξ,ζ are defined such that $u(s,t)=\exp_{y(t)}(\xi(s,t))$ for $s\geq\rho_0-1$ and $v(s,t)=\exp_{y(t)}(\zeta(s,t))$ for $s\leq -\rho_0+1$; $\beta\in C^\infty(\mathbb{R},[0,1])$ is non-decreasing such that $\beta(s)=0$ for $s\leq 0$ and $\beta(s)=1$ for $s\geq 1$. One then uses the Picard's method (see [8] lemma 4.2) to prove the existence of a vector field $\xi=\xi(u,v,\rho)$ on $w := u\hat{\#}_\rho v$ with $||\xi||_{W^{1,p}} \leq e^{-c\rho}$ and such that $\exp_w\xi$ is in $\mathcal{M}(x,z)$. We then define $\hat{u}\#_\rho\hat{v} := [\exp_w\xi] \in \hat{\mathcal{M}}(x,z)$. Of course this also gives the parametrized version of the gluing of u and v, and up to reparametrizations of u, v and $u\#_\rho v$, $u\#_\rho v$ is uniquely defined by u and v.

The techniques of gluing combined with Gromov's compactness are useful to prove several important results as we will see later.

4.3 Floer Homology

We will use the Maslov index to give a grading of the Floer homology groups. Proposition 4.8 is used to construct the boundary operator in Floer homology. Let (H, J) be a fixed regular pair. For simplicity, we work with $\mathbb{Z}_2$ coefficient only. The readers are reminded of the many similarities between the construction of Floer homology and that of Morse homology as given in section 1.2.

Definition 4.10. *Define the* ***k-th Floer chain complex*** *as the vector space over* $\mathbb{Z}_2$ *generated by the periodic solution* $x \in P(H)$ *of* (2.1) *with Maslov index* k

$$C_k := \mathrm{span}_{\mathbb{Z}_2}\{x \in P(H) : \mu(x) = k\}.$$

If $\mu(x) - \mu(y) = 1$*, define* $\langle \partial x, y\rangle := \#\hat{\mathcal{M}}(x, y)$ *(mod 2). Then the boundary operator* $\partial_k : C_k \to C_{k-1}$ *is defined by*

$$\partial_k x := \sum_{y \in C_{k-1}} \langle \partial x, y\rangle y$$

for $x \in P(H)$ *with* $\mu(x) = k$ *and extends it linearly.*

Floer proved the following theorem in the monotone case, establishing the existence of Floer homology:

Theorem 4.11. *(Floer [11]) The boundary operators satisfy*

$$\partial_k \circ \partial_{k+1} = 0.$$

Proof. Fix $x \in C_{k+1}$, this statement is equivalent to

$$\sum_{y \in C_k, z \in C_{k-1}} \langle \partial x, y \rangle \langle \partial y, z \rangle z = 0 \pmod 2.$$

So it is equivalent to prove for $x \in C_{k+1}, z \in C_{k-1}$, the number of pairs of unparametrized trajectories $(\hat{u}, \hat{v}) \in \hat{\mathcal{M}}(x, y) \times \hat{\mathcal{M}}(y, z)$ with $y \in C_k$ is even. This is proved by analyzing $\hat{\mathcal{M}}(x, z)$. First note that any component of the one-dimensional manifold $\hat{\mathcal{M}}(x, z)$ can only be a circle or is an open interval by the classification theorem. By (3) in the gluing proposition 4.9, each of the above $(\hat{u}, \hat{v})$ corresponds to an endpoint of a non-compact component (i.e. an open interval) of $\hat{\mathcal{M}}(x, z)$. By Gromov's compactness theorem 4.3, the other endpoint of this component must converge geometrically to another pair of unparametrized trajectories connecting x and z. Therefore there must be an even number of such pairs. □

Definition 4.12. *Define the* ***Floer homology groups*** *of the pair* (H, J) *on* M

$$HF_k(M; H, J) := \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

4.4 Invariance of Floer Homology

We will show in this section that the Floer homology groups are independent of the choice of the regular pair (H, J).

Note that the space $\mathcal{J}$ of all ω-compatible almost complex structures is contractible, thus we can always find a smooth homotopy between two regular pairs (H^α, J^α) and (H^β, J^β). More precisely, this consists of a smooth homotopy of Hamiltonian functions $H^{\beta\alpha} : \mathbb{R} \times \mathbb{S}^1 \times M \to \mathbb{R}$ and a smooth homotopy of almost complex structures $J^{\beta\alpha} : \mathbb{R} \times M \to C^\infty(\mathrm{End(TM)})$ such that

$$(H^{\beta\alpha}(s,t,x), J^{\beta\alpha}(s,x)) = \begin{cases} (H^\alpha(t,x), J^\alpha(x)) & \text{if } s \leq -T \\ (H^\beta(t,x), J^\beta(x)) & \text{if } s \geq T \end{cases}$$

for some $T > 0$.

Given such homotopies and suppose $x^\alpha \in P(H^\alpha)$, $x^\beta \in P(H^\beta)$. Denote $H^{\beta\alpha}$ by H and $J^{\beta\alpha}$ by J. Consider the solution $u : \mathbb{R} \times \mathbb{S}^1 \to M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(s,u)\frac{\partial u}{\partial t} + \nabla H(s,t,u) = 0 \tag{4.11}$$

with the boundary conditions

$$\lim_{s \to -\infty} u(s,t) = x^\alpha(t), \quad \lim_{s \to \infty} u(s,t) = x^\beta(t). \tag{4.12}$$

This equation can be considered as the gradient flow equation of the time dependent action functional $A = A_{H_s}$ with respect to the time-dependent metric induced by J_s. As in theorem 2.14, the solutions

of (4.11) with (4.12) has bounded energy and conversely a solution of (4.11) with bounded energy implies the limits in (4.12) exist.
As before, for such u, by linearizing (4.11) in the direction of $\xi \in C^\infty(u^*TM)$, we get the operator $F(u) : W^{1,p}(u^*TM) \to L^p(u^*TM)$ for $p \geq 2$ defined by

$$F(u)\xi := \nabla_s \xi + J(s,u)\nabla_t \xi + (\nabla_\xi J(s,u))\frac{\partial u}{\partial t} + \nabla_\xi \nabla H(s,t,u) \quad (4.13)$$

All the previous results still hold in this time dependent case. In particular $F(u)$ is a Fredholm operator and its index is given by

$$\operatorname{index} F(u) = \mu(x^\alpha, H^\alpha) - \mu(x^\beta, H^\beta).$$

Definition 4.13. *Let (H^α, J^α), (H^β, J^β) be two regular pairs, a smooth homotopy $(H^{\beta\alpha}, J^{\beta\alpha})$ between them is called a **regular homotopy** if for any $x^\alpha \in P(H^\alpha)$, $x^\beta \in P(H^\beta)$, whenever u satisfies* (4.11) *and* (4.12), *$F(u)$ is onto.*

The space of all regular homotopies is a dense subset in the space of all homotopies between (H^α, J^α) and (H^β, J^β) in C^∞_{loc} topology, i.e. in the topology of uniform convergence of all derivatives on compact subset. For such a regular homotopy, the space

$$\mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha}) := \{u : \mathbb{R}\times\mathbb{S}^1 \to M \mid u \text{ satisfies (4.11) and (4.12)}\}$$

of connecting trajectories is a finite dimensional manifold of dimension

$$\dim \mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha}) = \mu(x^\alpha, H^\alpha) - \mu(x^\beta, H^\beta).$$

Note that in this case the shifting property by $\mathbb{R}$ on $\mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha})$ is lost. $\mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha})$ can be non-empty even if $\mu(x^\alpha, H^\alpha) = \mu(x^\beta, H^\beta)$. In analogy with proposition 4.8 the 0-dimensional manifold $\mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha})$ is compact when $\mu(x^\alpha, H^\alpha) = \mu(x^\beta, H^\beta)$, i.e. it is finite. In this case, we define the number

$$\langle \phi^{\beta\alpha} x^\alpha, x^\beta \rangle := \#\mathcal{M}(x^\alpha, x^\beta; H^{\beta\alpha}, J^{\beta\alpha}) \pmod 2.$$

We then define the map

$$\phi^{\beta\alpha} = \phi(H^{\beta\alpha}, J^{\beta\alpha}) : C_k(M; H^\alpha) \to C_k(M; H^\beta)$$

by

$$\phi^{\beta\alpha} x^\alpha := \sum_{\mu(x^\beta, H^\beta) = k} \langle \phi^{\beta\alpha} x^\alpha, x^\beta \rangle x^\beta$$

if $\mu(x^\alpha, H^\alpha) = k$.

Proposition 4.14. *For a regular homotopy $(H^{\beta\alpha}, J^{\beta\alpha})$ between two regular pairs (H^α, J^α) and (H^β, J^β), the map $\phi = \phi^{\beta\alpha}$ as constructed above is a chain map, i.e.*

$$\phi \circ \partial^\alpha = \partial^\beta \circ \phi.$$

Sketch of proof. The proof is similar to that of theorem 4.11. Denote $C_k^\alpha := C_k(M; H^\alpha)$ and $C_k^\beta := C_k(M; H^\beta)$. We have to show for any $x \in C_{k+1}^\alpha$ and $y \in C_k^\beta$,

$$\sum_{z \in C_k^\alpha} \langle \partial^\alpha x, z \rangle \langle \phi z, y \rangle = \sum_{w \in C_{k+1}^\beta} \langle \phi x, w \rangle \langle \partial^\beta w, y \rangle \pmod 2$$

It suffices to show that there is an even number of pairs of unparametrized trajectories between x and y. Let $(\hat{u}, v) \in \hat{\mathcal{M}}(x, z; H^\alpha, J^\alpha) \times \mathcal{M}(z, y; H^{\beta\alpha}, J^{\beta\alpha})$, where $z \in C_k^\alpha$. As in the proof of theorem 4.11, each such $(\hat{u}, v) \in \hat{\mathcal{M}}(x, z; H^\alpha, J^\alpha) \times \mathcal{M}(z, y; H^{\beta\alpha}, J^{\beta\alpha})$ corresponds to an endpoint of a non-compact component (i.e. an open interval) of the one-dimensional manifold $\mathcal{M}(x, y; H^{\beta\alpha}, J^{\beta\alpha})$. One endpoint of this component is identified with $(\hat{u}, v)$ as above, and the other endpoint must be identified with a pair $(q, \hat{r}) \in \mathcal{M}(x, w; H^{\beta\alpha}, J^{\beta\alpha}) \times \hat{\mathcal{M}}(w, y; H^\beta, J^\beta)$ with $w \in C_{k+1}^\beta$. Therefore there is an even number of pairs of unparametrized trajectories between x and y. □

Thus every regular homotopy $(H^{\beta\alpha}, J^{\beta\alpha})$ between (H^α, J^α) and (H^β, J^β) induces a homomorphism of Floer homology groups. This homomorphism turns out to be independent of the choice of the regular homotopy.

Proposition 4.15. *For two regular homotopies $(H_0^{\beta\alpha}, J_0^{\beta\alpha})$ and $(H_1^{\beta\alpha}, J_1^{\beta\alpha})$ from (H^α, J^α) to (H^β, J^β), the associated chain homomorphism $\phi_0 = \phi(H_0^{\beta\alpha}, J_0^{\beta\alpha})$ and $\phi_1 = \phi(H_1^{\beta\alpha}, J_1^{\beta\alpha})$ are chain homotopy equivalent. i.e. there exists $\Psi = \Psi_k : C_k^\alpha \to C_{k+1}^\beta$ such that*

$$\phi_1 - \phi_0 = \partial^\beta \circ \Psi + \Psi \circ \partial^\alpha. \tag{4.14}$$

Sketch of proof. Again this proof is similar to that of theorem 4.11, as it also uses the compactness-gluing argument. Define C_k^α and C_k^β

as before and $(H_i, J_i) := (H_i^{\beta\alpha}, J_i^{\beta\alpha})$ for $i = 0, 1$. We will define the chain homotopies $\Psi = \Psi_k : C_k^\alpha \to C_{k+1}^\beta$ such that

$$\phi_1 - \phi_0 = \partial^\beta \circ \Psi + \Psi \circ \partial^\alpha. \tag{4.15}$$

(The sign is not important as we are working in $\mathbb{Z}_2$ coefficient.) To do this, choose a one-parameter family of of homotopy $\lambda \mapsto (H_\lambda, J_\lambda)$ connecting (H_0, J_0) and (H_1, J_1), $\lambda \in [0, 1]$, and let $x \in P(H^\alpha)$, $y \in P(H^\beta)$. ((H_λ, J_λ) may not be a regular homotopy if $\lambda \neq 0, 1$.) Define the λ-parametrized moduli space

$$\tilde{\mathcal{M}}(x, y) := \{(\lambda, u) : \lambda \in [0, 1], u \in \mathcal{M}(x, y; H_\lambda, J_\lambda)\}.$$

Then by a similar analysis as before, for a generic choice of (H_λ, J_λ), this space is a finite dimensional manifold with boundary, and

$$\dim \tilde{\mathcal{M}}(x, y) = \mu(x, H^\alpha) - \mu(y, H^\beta) + 1. \tag{4.16}$$

(The parameter λ gives one more dimension than those not parametrized by λ.) Suppose now $x \in C_k^\alpha$, $y \in C_{k+1}^\beta$, so $\tilde{\mathcal{M}}(x, y)$ is zero dimensional, we claim that it is finite. Suppose not, then by the Gromov's compactness theorem 4.3, there are sequences $(\lambda_n, u_n) \in \tilde{\mathcal{M}}(x, y)$ with $\lambda_n \to \lambda_0$, $u_n \to u$. Then u satisfies the equation

$$\frac{\partial u}{\partial s} + J_{\lambda_0}(s, u)\frac{\partial u}{\partial t} + \nabla H_{\lambda_0}(s, t, u) = 0$$

for that particular λ_0. As u has bounded energy, $u \in \mathcal{M}(z, w)$ for some $z \in P(H^\alpha)$, $w \in P(H^\beta)$. Indeed we must have $u \in \mathcal{M}(x, y)$.

For otherwise by theorem 4.3, $\mu(z) < \mu(x)$ or $\mu(w) > \mu(y)$, but then $\tilde{\mathcal{M}}(z, w) = \emptyset$ by equation (4.16), a contradiction. So $(\lambda, u) \in \tilde{M}(x, y)$ is a cluster point of the zero dimensional $\tilde{M}(x, y)$, but this contradicts the manifold structure of $\tilde{\mathcal{M}}(x, y)$. Therefore $\tilde{\mathcal{M}}(x, y)$ must be finite.

We then define

$$\Psi(x) := \sum_{y \in C^{\beta}_{k+1}} \langle \Psi x, y \rangle y \pmod 2$$

where

$$\langle \Psi x, y \rangle := \# \tilde{\mathcal{M}}(x, y).$$

As before, proving (4.15) is equivalent to prove for $x \in C^{\alpha}_{k}$, $z \in C^{\beta}_{k}$,

$$\sum_{y \in C^{\beta}_{k+1}} \langle \Psi x, y \rangle \langle \partial^{\beta} y, z \rangle + \sum_{w \in C^{\alpha}_{k-1}} \langle \partial^{\alpha} x, y \rangle \langle \Psi w, z \rangle + \langle \phi_0 x, z \rangle + \langle \phi_1 x, z \rangle$$
$$= 0 \pmod 2. \tag{4.17}$$

This time we analyze $\tilde{\mathcal{M}}(x, z)$, we will show each boundary point and endpoint (which is not contained in $\tilde{\mathcal{M}}(x, z)$) of the precompact 1-dimensional manifold $\tilde{\mathcal{M}}(x, z)$ exactly contributes to one of the four factors on the L.H.S. of (4.17) (modulo two). As the total number of boundary points and endpoints are even, the theorem is then proved. There are four cases. For the first case, suppose the boundary point $(0, u) \in \tilde{\mathcal{M}}(x, y)$, then u is a point in $\mathcal{M}(x, z; H_0, J_0)$, so it contributes to $\langle \phi_0 x, z \rangle$. The second case where $(1, u) \in \tilde{\mathcal{M}}(x, z)$ is also similar and corresponds to an entry in $\langle \phi_1 x, z \rangle$.

For the remaining cases, suppose there exists a sequence $(\lambda_n, u_n) \in \tilde{\mathcal{M}}(x,z)$ with $(\lambda_n, u_n) \to (\lambda_0, u) \notin \tilde{\mathcal{M}}(x,z)$. Then u is a bounded solution of (4.11) with $(H,J) = (H_{\lambda_0}, J_{\lambda_0})$. For generic (H_λ, J_λ), if $\mu(x, H^\alpha) = \mu(z, H^\beta) = k$, there are only two possibilities, either $u \in \mathcal{M}(x, y; H_{\lambda_0}, J_{\lambda_0})$ for some $y \in C^\beta_{k+1}$ or $u \in \mathcal{M}(w, z; H_{\lambda_0}, J_{\lambda_0})$ for some $w \in C^\alpha_{k-1}$. In the first case, by compactness, u_n must converges geometrically (see remark 4.4) to a pair (u, v) where $v \in \mathcal{M}(y, z; H^\beta_{\lambda_0}, J^\beta_{\lambda_0})$, and in the second case u_n converges geometrically to a pair (r, u) where $r \in \mathcal{M}(x, w; H^\alpha_{\lambda_0}, J^\alpha_{\lambda_0})$, and both the pair (u, v) and (r, u) corresponds to an endpoint of a non-compact component of $\tilde{\mathcal{M}}(x,z)$ by the gluing argument. This proves our claim. $\square$

Proposition 4.15 shows that there exists a homomorphism of Floer homology groups which we still denote by

$$\phi^{\beta\alpha} : HF_*(M; H^\alpha, J^\alpha) \to HF_*(M; H^\beta, J^\beta).$$

Theorem 4.16. *Let (H^α, J^α), (H^β, J^β) and (H^γ, J^γ) be regular pairs. Then*

$$\phi^{\beta\alpha} : HF_*(M; H^\alpha, J^\alpha) \xrightarrow{\cong} HF_*(M; H^\beta, J^\beta)$$

is an isomorphism and

$$\phi^{\gamma\beta} \circ \phi^{\beta\alpha} = \phi^{\gamma\alpha}, \quad \phi^{\alpha\alpha} = \mathrm{id}. \tag{4.18}$$

Sketch of proof. It suffices to prove relations (4.18) hold, as $\phi^{\beta\alpha}$ must be an isomorphism with inverse $\phi^{\alpha\beta}$ by choosing $\gamma = \alpha$.

$\phi^{\alpha\alpha} = \mathrm{id}$ follows by choosing the constant homotopy.

Now given $(H^{\beta\alpha}, J^{\beta\alpha})$ and $(H^{\gamma\beta}, J^{\gamma\beta})$ being regular homotopies from (H^α, J^α) to (H^β, J^β) and from (H^β, J^β) to (H^γ, J^γ) respectively. Then for $R > 0$ large enough,

$$(H_R(s,t,x), J_R(s,x)) := \begin{cases} (H^{\beta\alpha}(s+R,t,x), J^{\beta\alpha}(s+R,x)) & \text{if } s \leq 0 \\ (H^{\gamma\beta}(s-R,t,x), J^{\gamma\beta}(s-R,x)) & \text{if } s \geq 0. \end{cases}$$

is a regular homotopy from (H^α, J^α) to (H^γ, J^γ). Denote by $\phi_R : C_k^\alpha \to C_k^\gamma$ the associated chain homomorphism, using the notation as in the proof of proposition 4.14. It suffices to prove

$$\phi_R = \phi^{\gamma\beta} \circ \phi^{\beta\alpha},$$

or equivalently for any fixed $x \in C_k^\alpha$, $z \in C_k^\gamma$,

$$\langle \phi_R x, z \rangle = \sum_{y \in C_k^\beta} \langle \phi^{\beta\alpha} x, y \rangle \langle \phi^{\gamma\beta} y, z \rangle z. \tag{4.19}$$

This again follows by a compactness-gluing argument.

Each pair $(u,v) \in \mathcal{M}(x,y; H^{\beta\alpha}, J^{\beta\alpha}) \times \mathcal{M}(y,z; H^{\gamma\beta}, J^{\gamma\beta})$ for $y \in C_k^\beta$ can be glued together to obtain $u_R \in \mathcal{M}(x,z; H_R, J_R)$ for sufficiently large R. (There are finitely many such pair, thus gives rise to a gluing map.) Conversely by (3) of proposition 4.9, for large enough R, any $u_R \in M(x,y; H_R, J_R)$ must lies within the range of this gluing map, i.e. there is no other trajectories in $\mathcal{M}(x,z; H_R, J_R)$. This implies (4.19). $\square$

4.5 An Isomorphism Theorem

We will show in this section that the Floer homology groups are isomorphic to the singular homology groups of M up to a shift of the grading. We mainly follow the approach in [33].

Although we are mainly interested in 1-periodic solutions of (2.1), to establish the above result we have to consider also the τ-periodic solutions for arbitrary τ and we denote this solutions set to be $P_\tau(H)$. We define $\mathcal{M}(x, y; \tau)$ for $x, y \in P_\tau(H)$ and $F_\tau(u)$ for $u \in \mathcal{M}(x, y; \tau)$ as in (3.4) analogously as before. We consider an Morse function $H(t, x) = H(x)$ on M which is independent of t. Then there exists $\varepsilon > 0$ such that every non-constant periodic solution of (2.1) is of period greater than ε. In other words if $\tau < \varepsilon$ then the τ-periodic solutions are exactly critical points of H:

$$P_\tau(H) = \{x(t) \equiv x \in M : dH(x) = 0\}.$$

For a constant periodic solution x, it turns out that the Maslov index of x regarded as a 1-periodic solution is related to its Morse index $\lambda(x, H)$ regarded as a critical point. More precisely, fix a Riemannian metric on M by choosing a compatible almost complex structure J on M, we have

Lemma 4.17. *There exists $\varepsilon > 0$ such that for any Morse function $H : M \to \mathbb{R}$ with $||H||_{C^2} < \epsilon$ and for every critical point x of H,*

$$\mu(x, H) = \lambda(x, H) - n.$$

For the proof, we need another lemma first.

Lemma 4.18. *Suppose p is a critical point of a Morse function $H : M \to \mathbb{R}$, $\{x_1, \cdots, x_{2n}\}$ are local coordinates around p such that J is represented by J_0 and $\frac{\partial}{\partial x_i}$ forms an orthonormal basis for T_pM, then the differential $d\psi_t(p)$ of ψ_t at p is given by*

$$d\psi_t(p) = e^{tJ_0 d^2 H(p)}$$

where $d^2H(p)$ denotes the Hessian of H at p.

Proof. We denote the differential with respect to $x \in \mathbb{R}^{2n}$ by d. Then in the given local coordinates,

$$\begin{aligned} \frac{d}{dt}\psi_t &= J_0 \nabla H(\psi_t) \\ \Rightarrow d(\frac{d}{dt}\psi_t) &= d(J_0 \nabla H(\psi_t)) \\ \frac{d}{dt}(d\psi_t) &= J_0(d\nabla H)(d\psi_t) = J_0(d^2H)(d\psi_t). \end{aligned}$$

So $d\psi_t(x)$ satisfies the following linear system of O.D.E.

$$\begin{cases} \frac{d}{dt}\Phi(t,x) = J_0(d^2H)\Phi(t,x) \\ \Phi(0,x) = I_{2n\times 2n}. \end{cases}$$

By the uniqueness of solution, since $e^{J_0(d^2H)t}$ is also a solution, so at p,

$$d\psi_t(p) = e^{tJ_0 d^2 H(p)}.$$

□

Proof of lemma 4.17. Choose a symplectic orthonormal basis for T_xM such that J is represented by $J_0 \in M(2n;\mathbb{R})$ in this basis. Let $S \in M(2n;\mathbb{R})$ represents the Hessian matrix of H at x with respect to this basis. By lemma 4.18, the symplectic path induced by ψ_t is then given by

$$\Psi(t) = \exp(J_0 S t), \quad t \in [0,1].$$

S is non-singular as H is a Morse function, so $\lambda(x, H)$ is given by the number of negative eigenvalues $\lambda(S)$ of S. On the other hand, by definition, $\mu(x) = \mu(\Psi)$. By theorem 3.7, $\mu(\Psi) = \lambda(S) - n$. Therefore $\mu(x) = \lambda(x, H) - n$. □

Observe that if $H(t, x) = H(x)$ is independent of t, then for those solutions $u = u(s)$ to (2.7) which is also independent of t satisfies

$$\frac{du}{ds} = -\nabla H(u). \tag{4.20}$$

i.e. it satisfies the gradient flow equation for H. Recall that the system (4.20) is said to satisfy the **Morse-Smale condition** if for any two critical points x and y of H, the unstable manifold $W^u(x)$ and the stable manifold $W^s(y)$ intersect transversally.

The Morse homology theorem (theorem 1.24) states that if (4.20) satisfies the Morse-Smale condition, then the Morse homology of H is isomorphic to the singular homology of M (g is the metric induced by J):

$$HM_k(M; H, g) \cong H_k(M; \mathbb{Z}_2).$$

Proposition 4.19. *Let J be an almost complex structure on M compatible with ω and $H : M \to \mathbb{R}$ is a Morse function such that* (4.20) *is Morse-Smale. Then for sufficiently small $\tau > 0$,*

1. *if $u : \mathbb{R} \to M$ is a flow line of* (4.20)*, then $F_\tau(u)$ is surjective, and*

2. *if $u(s,t) = u(s,t+\tau)$ is a bounded τ-periodic solution of* (2.7)*, then u is independent of t.*

See [30] for the proof of (1) and [33], [30] for the proof of (2).

Theorem 4.20. *Let (H, J) be a regular pair. Then there exists an isomorphism*

$$HF_k(M; H, J) \to H_{k+n}(M; \mathbb{Z}_2)$$

from the Floer homology of the pair (H, J) to the singular homology of M.

Proof of theorem 4.20. We use the notation $HF_k(M; H, J, \tau)$ to denote the Floer homology as defined before except that the complex is taken from $P_\tau(H)$. In view of theorem 4.16 it suffices to prove this theorem for any Morse function $H : M \to \mathbb{R}$ and $J \in \mathcal{J}$ such that (4.20) is a Morse-Smale flow. By proposition 4.19, (H, J) is regular for sufficiently small $\tau > 0$ and the τ-periodic solutions of (4.11) and (4.12) are independent of t. Thus by lemma 4.17, the Morse complex of the gradient flow (4.20) agrees with the Floer complex with a shifting of n and so we have

$$HF_k(M; H, J, \tau) \cong HM_{k+n}(M; H, g) \cong H_{k+n}(M; \mathbb{Z}_2).$$

Then observe that the Floer homology groups are independent of the choice of τ. For a solution $u(s,t) = u(s, t+\tau)$ of (4.20) with respect to a τ-periodic Hamiltonian $H(t,x)$, $v(s,t) := u(\tau s, \tau t)$ is a 1-periodic solution of the corresponding equation with respect to the 1-periodic Hamiltonian function

$$H_1(t,x) := \tau H(\tau t, x).$$

Also, the corresponding periodic solutions have the same Maslov index and therefore

$$HF_k(M; H_1, J, 1) \cong HF_k(M; H, J, \tau) \cong H_{k+n}(M; \mathbb{Z}_2).$$

$\square$

So applying the weak Morse inequality (theorem 1.25), Floer [11] proved the Arnold conjecture in the monotone case.

Corollary 4.21 (Arnold conjecture)**.** *Suppose all the periodic solutions of* (2.1) *are non-degenerate, then the number of periodic solutions to* (2.1) *is bounded below by the sum of Betti numbers b_i's of M:*

$$\#P(H) \geq \sum_{i=0}^{2n} b_i.$$

4.6 Further Applications

We will give in this section some further applications of Floer's theory. [22] is a good reference and gives a more complete picture than what I give here.

Floer's theory for Lagrangian intersection.

As remarked in remark 2.5, the fixed points of a Hamiltonian symplectomorphism can be regarded as the points of intersection of the graph Γ of ψ and the diagonal Δ in $M \times M$. This can be regarded as a special case of the intersection of two Lagrangian submanifolds (submanifolds of maximal dimension where the symplectic form restricts to zero): if the product $M \times M$ is endowed with the symplectic form $\omega \times -\omega$, then it can be showed that both Γ and Δ are Lagrangian, and Γ is the image of Δ under the Hamiltonian symplectomorphism $\mathrm{id} \times \psi$. Thus Arnold conjecture can be generalized to ask for the minimal number of points of intersection of a Lagrangian submanifold L and its image $\psi(L)$ under a Hamiltonian symplectomorphism ψ in a symplectic manifold M, provided that the intersection is transveral (this corresponds to the requirement that the fixed points are non-degenerate in chapter 2). Floer's theory can again be applied in this case and the Morse inequality as given in corollary 4.21 was proved by Floer [8] under the assumption that ω vanishes on $\pi_2(M, L)$.

Morse theory for periodic solutions

By considering the Floer homology groups with the chain groups being the contractible τ-periodic solutions to (2.1), τ can be different from 1 (as in section 4.5), Salamon and Zehnder [33] show that under assumption 2.10, and that if all the contractible 1-periodic solutions to (2.1) are weakly non-degenerate, then there are infinitely many contractible periodic solutions with integer periods.

Here a periodic solution x is weakly non-degenerate if at least one eigenvalue of $d\psi(x(0))$ is not 1.

Similar to the proof of Arnold's conjecture, by applying the Morse inequality (theorem 1.26), denoting the number of contractible τ-periodic solutions to (2.1) with Maslov index k by $p_k(\tau)$, they also obtain the following Morse-type inequality:

Theorem 4.22. *Suppose all the periodic solutions of* (2.1) *are non-degenerate, then*

$$b_{n+k} - b_{n+k-1} + \cdots (-1)^{n+k} b_0 \leq p_k(\tau) - p_{k-1}(\tau) + \cdots$$

for $k \in \mathbb{Z}$, where b_k's denote the Betti numbers of M.

Bibliography

[1] V. Arnold. Sur une propriété topologique des applications globalement canoniques de la mécanique classique. *CR Acad. Sci. Paris*, 261:3719–3722, 1965.

[2] V. Arnold. Mathematical methods in classical mechanics,("Nauka", Moscow), 1974. *English transl.(Springer-Verlag)*, 1978.

[3] M. Atiyah, V. Patodi, and I. Singer. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc*, 79(1):71–99, 1976.

[4] R. Bott. Morse theory indomitable. *Publications Mathématiques de L'IHÉS*, 68(1):99–114, 1988.

[5] C. Conley and E. Zehnder. Morse Type Index Theory for Flows and Periodic Solutions for Hamiltonian Equations. 1983.

[6] C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold. *Inventiones Mathematicae*, 73(1):33–49, 1983.

[7] A. Floer. A relative Morse index for the symplectic action. *Communications on pure and applied mathematics*, 41(4):393–407, 1988.

[8] A. Floer. Morse theory for Lagrangian intersections. *Journal of differential geometry*, 28(3):513–547, 1988.

[9] A. Floer. The unregularized gradient flow of the symplectic action. *Comm. Pure Appl. Math*, 41(6):775–813, 1988.

[10] A. Floer. Cuplength estimates on Lagrangian intersections. *Comm. Pure Appl. Math*, 42(4):335–356, 1989.

[11] A. Floer. Symplectic fixed points and holomorphic spheres. *Communications in Mathematical Physics*, 120(4):575–611, 1989.

[12] A. Floer. Witten's complex and infinite dimensional Morse theory. *J. Differential Geom*, 30(1):207–221, 1989.

[13] A. Floer, H. Hofer, and D. Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J*, 80(1):251–292, 1995.

[14] K. Fukaya and K. Ono. Arnold Conjecture and Gromov-Witten Invariant for General Symplectic Manifolds. *The Arnoldfest: Proceedings of a Conference in Honour of VI Arnold for His Sixtieth Birthday*, 24, 1999.

[15] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Inventiones Mathematicae*, 82(2):307–347, 1985.

[16] H. Hofer. Lusternik-Schnirelman-theory for Lagrangian intersections. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(5):465–499, 1988.

[17] H. Hofer and D. Salamon. Floer homology and Novikov rings. The Floer memorial volume, 483–524. *Progr. Math*, 133.

[18] M. Hutchings. Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves).

[19] J. Jost. *Riemannian geometry and geometric analysis.* Springer New York, 1998.

[20] S. Lang. *Introduction to Differentiable Manifolds.* Springer, 2002.

[21] G. Liu and G. Tian. Floer homology and Arnold conjecture. *J. Differential Geom*, 49:1–74, 1998.

[22] D. McDuff. Elliptic methods in symplectic geometry. *American Mathematical Society*, 23(2), 1990.

[23] D. McDuff and D. Salamon. Introduction to Symplectic Topology. *Oxford Science Publications*, 1998.

[24] D. McDuff and D. Salamon. *J-Holomorphic Curves and Symplectic Topology.* American Mathematical Society, 2004.

[25] J. Milnor. *Morse Theory*. Princeton University Press, 1963.

[26] K. Ono. On the Arnold conjecture for weakly monotone symplectic manifolds. *Inventiones Mathematicae*, 119(1):519–537, 1995.

[27] J. Palis and W. de Melo. *Geometric theory of dynamical systems. An introduction. Transl. from the Portuguese by AK Manning*. New York-Heidelberg-Berlin: Springer-Verlag, 1982.

[28] J. Robbin and D. Salamon. The spectral flow and the Maslov index. *Bull. London Math. Soc*, 27(1):1–33, 1995.

[29] Y. Ruan. Virtual neighborhoods and pseudo-holomorphic curves. *Proceedings of 6th Gökova Geometry-Topology Conference*, pages 161–231.

[30] D. Salamon. Morse theory, the Conley index and Floer homology. *Bull. London Math. Soc*, 22(2):113–140, 1990.

[31] D. Salamon. Lectures on Floer homology. *Symplectic Geometry and Topology*, 7:143–229, 1999.

[32] D. Salamon and E. Zehnder. Floer homology, the Maslov index and periodic orbits of Hamiltonian equations. *Analysis Et Cetera (PH Rabinowitz and E. Zender, eds.), Academic Press*, pages 573–600, 1990.

[33] D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math*, 45(10):1303–1360, 1992.

[34] M. Schwarz. *Morse homology.* Basel; Boston: Birkhäuser Verlag, 1993.

[35] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math*, 87(4):861–866, 1965.

[36] S. Smale. Stable manifolds for differential equations and diffeomorphisms. *The Collected Papers of Stephen Smale*, 2000.

[37] A. Weinstein. Periodic orbits for convex Hamiltonian systems. *Ann. of Math*, 108(2):507–518, 1978.

[38] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17(4):661–692, 1982.