Report of An International Summer School on Geometric Topology 2006, Dalian

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The International Summer school in geometric topology 2006 in Dalian consists of ten lectures, introducing the materials and tools used in geometric topology, both basic and advanced, including basic 3-manifolds theory, surfaces in 3-manifolds, volume conjecture, variational principles on triangulated surfaces, transformation groups, contact 3-manifolds, quantum Teichmuller space, Heegaard Floer homology and knot invariants, etc. Due to the vast amount of materials and the lack of familiarities with most topics, here I will only write something in the most basic 3-manifolds theory. Even with this relatively narrow scope, this is by no means a survey of this topic, nor are the following contents closely connected with each others.

We begin with some definitions.

Definition 1 *A topological n-manifold is a separable metric space each point of which has an open neighborhood homeomorphic to either* $\mathbb{R}^n$ *or* $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$.

The piecewise linear (PL) point of view can be useful to study 3-manifolds, partly because of the "finiteness" of polyhedra, partly because of the ease of visualization of low dimensional polyhedra, and also the following theorem 1 due to Moise and Bing. I assume the definitions of simplicial complex K, subdivision of K, its underlying space $|K|$ and a triangulation of a space (T, h), where T is a simplicial complex and h is a homeomorphism from $|T|$ to X, are well understood.

A map between underlying space of simplicial complexes $f : |K_1| \to |K_2|$ is piecewise linear (PL) if there exist subdivisions L_1 of K_1 and L_2 of K_2 w.r.t. f is simplicial i.e. f take vertices of L_1 to vertices of L_2 and take each simplex of L_1 linearly in barycentric coordinates onto a simplex of L_2.

Definition 2 *A triangulation* (T, h) *of a space* X *is a simplicial complex* T *with a homeomorphism* $h : |T| \to X$.
(T_1, h_1) *and* (T_2, h_2) *are compatible if* $h_2^{-1}h_1$ *is PL.*

Definition 3 *A PL homeomorphism is a homeomorphism f such that f and f^{-1} is PL.*

Definition 4 *A PL structure on manifold M is a maximal nonempty collection of compatible combinatorial triangulation of M. (Combinatorial means a certain combinatoric condition concerning with a triangulation, its definition is omitted here.)*

Definition 5 *A PL manifold is a manifold M with a PL structure on it.*

Definition 6 *A PL n-cell, denoted by B^n or $\mathbb{D}^n$, is a PL manifold PL homeomorphic to a n-simplex Δ^n. A PL n-sphere is a PL manifold PL homeomorphic to $\partial\Delta^{n+1}$.*

There can be many more definitions in the PL setting, such as PL submanifold, orientation etc, which are trivial to state and are similar to their analogues in the differentiable setting. The importance of the PL category is due to:

Theorem 1 (Moise, Bing) *Each topological 3-manifold has a PL structure unique up to PL homeomorphism.*

This theorem enables us to work entirely within PL category. There are also some elementary theorems:

Theorem 2 (Alexander) *Every PL 2-sphere in $\mathbb{R}^3$ bounds a PL 3-cell.*

Theorem 3 (Guggenheim) *Every orientation preserving PL homeomorphism of a PL n-cell or PL n-sphere onto itself is PL isotopic to the identity.*

Theorem 4 (Newman) *If D is a PL n-cell, then any PL homeomorphism of ∂D to itself can be extended to a PL homeomorphism of D itself.*

Theorem 5 (Newman) *If M is a PL n-manifold, and C is a PL n-cell such that $M \cap C = \partial M \cap \partial C$ is a Pl $(n-1)$-cell of M and C, then M is PL homeomorphic to $M \cup C$.*

From now on we will omit the word PL altogether.

We introduce the concept of connected sum, which intuitively is just gluing together two manifolds by deleting the interior of one disk from each manifold and attach the punctured manifold by gluing along the boundary of the disk. We will state a decomposition theorem which states that a 3-manifold can be uniquely decomposed into simpler components gluing together by connected sum.

Definition 7 (Connected Sum) *Let M_1, M_2 be connected 3-manifolds, $B_i \subset M_i$ are 3-cells and $R_i = M \setminus \overset{\circ}{B_i}$. Suppose there are embeddings $h_i : R_i \to M$ with $h_1(R_1) \cap h_2(R_2) = h_1(\partial B_1) = h_2(\partial B_2)$ and $M = h_1(R_1) \cup h_2(R_2)$. Then M is called a connected sum of M_1 and M_2, denoted as $M = M_1 \# M_2$. If M, M_1, M_2 are oriented we require $h_i : R_i \to M$ to be orientation preserving.*

Remark 1 *Up to homeomorphism, the operation is independent of the choice of B_i in M_i but depends on the orientation of M_i, i.e. there exist oriented 3-manifold M_1, M_2 with $M_1 \# M_2$ not homeomorphic to $M_1 \# (-M_2)$, where $-M_2$ denotes M_2 with the orientation reversed.*

It is not difficult to see that for any 3-manifold M, $M \# \mathbb{S}^3 \cong M$. In order to avoid this trivial decomposition (for the uniqueness of decomposition theorem), we have the concept of primes. Another related and important concept is that of an irreducible manifold.

Definition 8 *A 3-manifold M is prime if it is not homeomorphic to $\mathbb{S}^3$ and $M = M_1 \# M_2$ implies one of M_1, M_2 is a 3-sphere.*

Definition 9 *A 3-manifold M is irreducible if every 2-sphere in M bounds a 3-cell.*

Theorem 6 (Alexander) *$\mathbb{R}^3$ and $\mathbb{S}^3$ are irreducible.*

It is not difficult to see that

Theorem 7 *Every irreducible manifolds are prime.*

As a partial converse,

Theorem 8 *If M is a prime 3-manifold, then either it is irreducible or it is a 2-sphere bundle over $\mathbb{S}^1$.*

There is an important factorization theorem in 3-manifold theory:

Theorem 9 (Kneser, Milnor) *Every compact (oriented) 3-manifold can be uniquely expressed as a connected sum of a finite number of prime factors, i.e. $M = M_1 \# \ldots \# M_n$, where each M_i is prime. Furthermore, the decomposition is unique up to order and (oriented) homeomorphisms, i.e. $M = M_1 \# \ldots \# M_n = N_1 \# \ldots \# N_m$ are prime decompositions of M implies that m=n and after reordering, $M_i \cong N_i$.*

There are some very important theorems. One is Dehn's lemma, which roughly states that if there is a map of a disk into a 3-manifold M, which does not cross itself on the boundary of the disk, then there exists an embedding of the disk into M which coincides with the original map on the boundary of the disk. To be more precise,

Theorem 10 (Dehn's lemma) *If M is a 3-manifold and $f : \mathbb{D}^2 \to M$ is a map such that $f|_{\partial\mathbb{D}^2}$ is an embedding and $f^{-1}(f(\partial\mathbb{D}^2)) = \partial\mathbb{D}^2$.*

This theorem is important in the sense that it provides a tool to simplify any surface which crosses itself to one which does not. (However, although this theorem was first stated by Dehn, the first valid proof was given by Papakyriakopoulos about 50 years later.)

The loop and sphere theorems provide a link between algebra (homotopy theory) and geometry and are central to many results in 3-manifold theory.

Theorem 11 (Loop theorem) *Suppose M is a 3-manifold, S is a connected surface in ∂M and N is a normal subgroup of $\pi_1(S)$. Let $f : \mathbb{D}^2 \to M$ is a map such that $f(\partial\mathbb{D}^2) \subset S$ and $[f|_{\partial\mathbb{D}^2}] \notin N$, then there exists an embedding $g : \mathbb{D}^2 \to M$ such that $g(\partial\mathbb{D}^2) \subset S$ and $[g|_{\partial\mathbb{D}^2}] \notin N$.*

By setting $N = \{1\}$ in the previous theorem, one consequence is

Corollary 1 *With the same assumptions as the previous theorem, if the normal subgroup $ker\{\pi_1(S) \hookrightarrow \pi_1(M)\}$ is nontrivial, then a nontrivial element of this normal subgroup can be represented by a simple closed curve in S.*

Analogous to the loop theorem is the sphere theorem, which concerns $\pi_2(M)$ instead of $\pi_1(M)$.

Theorem 12 (Sphere theorem) *Suppose M is a orientable 3-manifold with nontrivial $\pi_2(M)$, then there exists an embedding $g : \mathbb{S}^2 \to M$ such that its image homotopy class represents a nontrivial element in $\pi_2(M)$.*

Embedding 2-manifolds (surfaces) into a 3-manifold is also a useful way to study the 3-manifold. One approach uses the Heegaard surface, that is to embed a surface into a 3-manifold to "cut open" it so that the components of its complement are as "simple" as possible. The other approach uses the incompressible surfaces, the idea is to embed a surface into a 3-manifold so that the surface is as "simple" as possible and carries both geometric and algebraic information.

The concept of Heegaard splitting is easier to state, although it may be less useful.

Definition 10 (Heegaard splitting) *Let M be a closed connected 3-manifold. Suppose V_1, V_2 be handlebodies of the same genus and $h : \partial V_1 \to \partial V_2$ is a homeomorphism such that M is the quotient space $V_1 \cup_h V_2$, then (V_1, V_2) is called s Heegaard splitting of M, their common boundary is called the Heegaard surface of the splitting.*

An important point is:

Theorem 13 *Each closed connected 3-manifold has a Heegaard splitting.*

Another notion is that of a compressible surface, which is perhaps more useful. Philosophically, an incompressible surface is a surface embedded in a 3-manifold which has been simplified as much as possible while remaining "nontrivial" inside the 3-manifold.

Definition 11 (Incompressible surface) *A surface F properly embedded in the 3-manifold M is compressible in M if either:*

1. *$F = \mathbb{S}^2$ and F bounds a 3-cell in M, or*
2. *there exists a disk $D \subset M$ (a compressing disk for F) such that $D \cap F = \partial D$ and $[\partial D]$ is not trivial in F.*

We say F is incompressible in M if it is not compressible in M.

Note that any properly embedded disk in M is incompressible in M.

Example 1 *Let H_n be a handlebody of genus g, then ∂H_n is compressible in H_n.*

There are a few sufficient conditions which ensure the existence of incompressible surfaces in 3-manifolds, both in the compact case and the non-compact case, which I skip here. Haken had a great contribution to the study of incompressible surfaces. He introduced the Haken manifold, and also proved that Haken manifolds have a certain hierarchy. He proposed the virtual Haken conjecture. However, the theory is too involved and technical for me, here I try my best to sketch some of the ideas.

Definition 12 *For a n-manifold M, a $(n-1)$-submanifold F is two-sided in M if there exists an embedding $h : F \times [-1, 1] \to M$ such that $h(x, 0) = x$, $forall x \in F$ and $h(F \times [-1, 1]) \cap \partial M = h(\partial F \times [-1, 1])$.*

Definition 13 *A 3-manifold is called a Haken manifold if it is compact, orientable, irreducible and contains a two-sided incompressible surface.*

Definition 14 *A 3-manifold finitely covered by a Haken manifold is said to be virtually Haken.*

The Virtually Haken conjecture asserts that every compact, irreducible 3-manifold with infinite fundamental group is virtually Haken.

Definition 15 *Let M be a Haken 3-manifold. A sequence $M = M_0 \supset M_1 \supset \ldots \supset M_n$ of 3-manifolds is a hierarchy of M if M_{i+1} is obtained from M_i by cutting open along an incompressible surface in M_i and M_n is a disjoint union of 3-cells. n is called the length of the hierarchy.*

There are some important results by Waldhausen:

Theorem 14 (Waldhausen) *Hierarchies exist for any Haken 3-manifold, and there exists a hierarchy of any Haken 3-manifold of length not greater than four.*

Theorem 15 *Suppose M_1, M_2 are closed Haken 3-manifolds such that $\pi_1(M_1) \cong \pi_1(M_2)$, then M_1 is homeomorphic to M_2.*

Theorem 16 (Thurston) *Haken 3-manifolds satisfy the geometrization conjecture (which perhaps has now become a theorem).*

References

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