Numerical Differential Equations
Homework 1

(Due: Apr. 16, 2007)

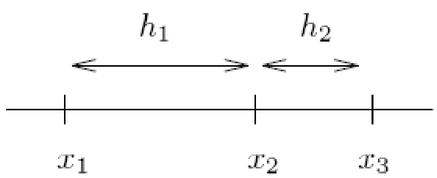
1. Let $u(x) = \sin x$ and $\bar{x} = 1$ and we are trying to approximate $u''(1) = -\sin 1$. Write two short MATLAB scripts implementing the following two second order approximations:

   - $$D^2u(x) = \frac{1}{h^2}[u(x+h) - 2u(x) + u(x-h)]$$

   - $$D_2^2u(x) = \frac{1}{4h^2}[u(x+2h) - 2u(x) + u(x-2h)]$$

   Use your MATLAB codes with various grid sizes (e.g., of the form 2^{-k}) to see whether the results of your numerical experiments correspond to the theory. Present your results both in graphical and table forms. Discuss the results.

2. Consider the nonuniform grid:



   (a) Use polynomial interpolation to derive a finite difference approximation for $u'(x_2)$ that is as accurate as possible for smooth functions u, based on the four values $U_1 = u(x_1)$, ... , $U_3 = u(x_3)$. Give an expression for the dominant term in the error.

   (b) Verify your expression for the error by testing your formula with a specific function and various values of h_1, h_2.

   (c) Can you define an "order of accuracy" for your method in terms of $h = max(h_1, h_2)$? To get a better feel for how the error behaves as the grid gets finer, do the following. Take a large number (say 500) of different values of H spanning two or three orders of magnitude, choose h_1 and h_2 as random numbers in the interval $[0, H]$ and compute the error in the resulting approximation. Plot these values against H on a log-log plot to get a scatter plot of the behavior as $H \to 0$. (Note: in matlab the command $h = H * rand(1)$ will produce a single random number uniformly distributed in the range $[0, H]$.) Of course these errors will not lie exactly on a straight line since the values of hk may vary quite a lot even for H's that are nearby, but you might expect the upper limit of the error to behave reasonably.

   (d) Estimate the "order of accuracy" by doing a least squares fit of the form

$$\log(E(H)) = K + p\log(H)$$