Numerical Differential Equations
Homework 3

(Due: Jul. 26, 2007)

Consider the initial value problem

$$y' = -\mu(y - t^2) + 2t, \quad 0 < t < T, \quad y(0) = y_0.$$

Recall that the exact solution is

$$y(t) = y_0 e^{-\mu t} + t^2.$$

Set $\mu = 520$, $T = 2$ and $y_0 = 4$.

A For this IVP, write (and present) three short MATLAB scripts implementing the following fixed step-size methods:

- the forward Euler method;
- the backward Euler method;
- the trapezoidal rule;

Note that because the right-hand side of the ODE is linear in y, you do not need to solve a nonlinear equation to obtain y_{n+1} from y_n in the two implicit schemes. Use this fact in your codes.

B Stability The "test equation" for the above ODE is

$$y' = -\mu y.$$

The results about the regions of absolute stability should give you an indication what the needed step-size to achieve absolute stability is for each of the above methods. Use your MATLAB codes with various time steps (e.g., of the form 2^{-k}) to see whether the results of your numerical experiments correspond to the theory. Present your results in a graphical form. Plot the exact and the numerical solution in the same picture. Also plot the global error e_n or its absolute value. Discuss the results.

As part of this problem, run your methods for the same equation on the interval $[a, b] = [1, 2]$ with the exact initial condition $y_0 = y(1) = 4e^{-520} + 1$. You should see that even though on this interval $[a, b]$ the solution is essentially t^2, the forward Euler method still requires a small step-size to get reasonable results. What about the other methods?

(As an illustrative example, you may want to change your codes so that they solve the equation $y' = 2t$ on the interval $[1, 2]$ with $y_0 = 1$. Observe that in this case the numerical methods do not exhibit any of the instability features.)

C Order of Convergence

For each of the four methods, establish numerically the order of convergence by computing the maximum of the global errors $|e^n|$ and observing its behavior as h converges to 0. Do the results correspond to the theory? Present the results in a graphical form or in a table.

For the trapezoidal rule, you should notice that in most of the interval [0,2] the global error is almost negligible. What is the order of the error? Can you explain why?