consider the following model initial-value problem:

$$\frac{d}{dt}u(t) = f(t)\,u(t), \quad t \in (0,T), \qquad u(0) = u_0, \qquad (1)$$

Suppose that we want to compute an approximation u_h to *u* on the interval (0*, T*) by using a DG method.

first find a partition $\{t^n\}_{n=0}^{N}$ of the interval (0*,T*) and set $I^n = (t^n, t^{n+1})$ for *n* = 0*, . . . , N* −1. Then we look for a function u_h which, on the interval I^n , is the polynomial of degree at most k^n determined by requiring that

$$-\int_{I^n} u_h(s)\,\frac{d}{dt}v(s)\,ds + \widehat{u}_h\,v\Big|_{t^n}^{t^{n+1}} = \int_{I_n} f(s)\,u(s)\,v(s)\,ds, \qquad (2)$$

for all polynomials v of degree at most k^n

To complete the definition of the DG method, we still need to define the quantity $\widehat{u_h}$.

Since for the ODE,the information travels "from the past into the future",
it is reasonable to take $\widehat{u_h}$ as follows:

$$\widehat{u}_h(t^n) = \begin{cases} u_0, & \text{if } t^n = 0, \\ \lim_{\epsilon \downarrow 0} u_h(t^n - \epsilon) & \text{otherwise.} \end{cases} \qquad (3)$$

This completes the definition of the DG method.

In this simple example, we already see the main components of the method,

(i) The use of *discontinuous* approximations u_h ,

(ii) The enforcing of the ODE on each interval by means of a Galerkin weak formulation, and

(iii) The introduction and suitable definition of the so-called *numerical trace*

The simple choice we have made is quite natural for this case and gives rise to a very good method; however, other choices can also produce excellent results. Next, we address the question of how to choose the numerical trace $\widehat{u_h}$

Let us begin with the problem of the consistency of the DG method.

As it is typical for most finite element methods, the method is said to be consistent if we can replace the approximate solution u_h by the exact solution *u* in the weak formulation (2).

We can immediately see that this is true if and only if $\hat{u}$ = *u*.

Next, let us consider the more subtle issue of the stability of the method.

Our strategy is to begin by obtaining a stability property for the ODE (1) which we will then try to enforce for the DG method (2) by a suitable definition of the numerical trace $\widehat{u_h}$.

If we multiply the ODE(1) by *u* and integrate over (0, *T*), we get the equality

$$\frac{d}{dt}u(t) = f(t)\,u(t), \quad t \in (0,T), \qquad \frac{1}{2}u^2(T) - \frac{1}{2}u_0^2 = \int_0^T f(s)\,u^2(s)\,ds,$$

set *v* = *uh* in the weak formulation (2), integrate by parts and add over *n*. We get

$$-\int_{I^n} u_h(s)\,\frac{d}{dt}v(s)\,ds + \widehat{u}_h\,v\Big|_{t^n}^{t^{n+1}} = \int_{I_n} f(s)\,u(s)\,v(s)\,ds,$$

$$\sum_{n=0}^{N-1}\left(-\frac{1}{2}u_h^2 + \widehat{u}_h\,u_h\right)\Bigg|_{t^n}^{t^{n+1}} = \frac{1}{2}u_h^2(T^-) + \Theta_h(T) - \frac{1}{2}u_0^2 = \int_0^T f(s)\,u_h^2(s)\,ds,$$

$$\tfrac{1}{2}\,u^2(T) - \tfrac{1}{2}\,u_0^2 = \int_0^T f(s)\,u^2(s)\,ds,$$

$$\sum_{n=0}^{N-1}\left(-\frac{1}{2}u_h^2 + \widehat{u}_h\,u_h\right)\Bigg|_{t^n}^{t^{n+1}} = \frac{1}{2}u_h^2(T^-) + \Theta_h(T) - \frac{1}{2}u_0^2 = \int_0^T f(s)\,u_h^2(s)\,ds,$$

where

$$\Theta_h(T) = -\frac{1}{2}u_h^2(T^-) + \sum_{n=0}^{N-1}\left(-\frac{1}{2}u_h^2 + \widehat{u}_h\,u_h\right)\Bigg|_{t^n}^{t^{n+1}} + \frac{1}{2}u_0^2.$$

$$\Theta_h(T) = -\frac{1}{2}u_h^2(T^-) + \sum_{n=0}^{N-1} \left(-\frac{1}{2}u_h^2 + \widehat{u}_h\, u_h \right)\Bigg|_{t^n}^{t^{n+1}} + \frac{1}{2}u_0^2.$$

Note that if $\Theta_h(T)$ were a non-negative quantity, the above equality would imply the stability of the DG method. In other words, the stability of the DG method is guaranteed if we can define the numerical trace $\widehat{u}_h$ so that $\Theta_h(T) \geq 0$. Setting

$$u_h(t) = u_0, \qquad t < 0,$$

and using the notation

$$\{u_h\} = \frac{1}{2}\left(u_h^- + u_h^+\right), \qquad [\![u_h]\!] = u_h^- - u_h^+, \qquad u_h^{\pm}(t) = \lim_{\epsilon \downarrow 0} u_h\left(t \pm \epsilon\right),$$

we rewrite $\Theta_h(T)$ as follows:

$$\begin{aligned}
\Theta_h(T) =& -\frac{1}{2}u_h^2(T^-) + \left(-\frac{1}{2}u_h^2(T^-) + \widehat{u}_h(T)\, u_h(T^-) \right) + \sum_{n=1}^{N-1} \left(-\frac{1}{2}[\![u_h^2]\!] + \widehat{u}_h\, [\![u_h]\!] \right)(t^n) \\
& - \left(-\frac{1}{2}u_h^2(0^+) + \widehat{u}_h(0)\, u_h(0^+) \right) + \frac{1}{2}u_0^2 \\
=& \left(\widehat{u}_h(T) - u_h(T^-)\right)\, u_h(T^-) + \sum_{n=1}^{N-1} \left((\widehat{u}_h - \{u_h\})\, [\![u_h]\!] \right)(t^n) - \left(\widehat{u}_h(0) - u_0\right)\, u_h(0^+) + \frac{1}{2}[\![u_h]\!]^2(0),
\end{aligned}$$

where we used the simple identity

$$[\![u_h^2]\!] = 2\,\{u_h\}\,[\![u_h]\!].$$

$$= \left(\widehat{u}_h(T) - u_h(T^-)\right) u_h(T^-) + \sum_{n=1}^{N-1} \left(\left(\widehat{u}_h - \{u_h\}\right) [\![u_h]\!]\right)(t^n) - \left(\widehat{u}_h(0) - u_0\right) u_h(0^+) + \frac{1}{2}[\![u_h]\!]^2(0),$$

where we used the simple identity

$$[\![u_h^2]\!] = 2\ \{u_h\}\ [\![u_h]\!].$$

It is now clear that if we take

$$\widehat{u}_h(t^n) = \begin{cases} u_0, & \text{if } t^n = 0, \\ \left(\{u_h\} + C^n\, [\![u_h]\!]\right)(t^n), & \text{if } t^n \in (0, T), \\ u_h(T^-), & \text{if } t^n = T, \end{cases}$$

where $C^n \geq 0$, we would have, setting $C^0 = 1/2$,

$$\Theta_h(T) = \sum_{n=0}^{N-1} C^n\, [\![u_h]\!]^2(t^n) \geq 0,$$

just as we wanted.

Note that the choice $C_n = 1/2$ corresponds to the numerical trace we chose at the beginning, namely,

Moreover, in our search for stability, we found, in a very natural way, that the numerical trace $\widehat{u_h}(t)$ can *only* depend on both traces of u_h at *t*, that is, on $u_h(t^-)$ and on $u_h(t^+)$.

Next, we want to emphasize three important properties of the DG methods that do carry over to the multi-dimensional case and to all types of problems.

The first is that the approximate solution of the DG methods does not have to satisfy any interelement continuity constraint.

As a consequence, the method can be highly parallelizable (when dealing with time-dependent hyperbolic problems) .

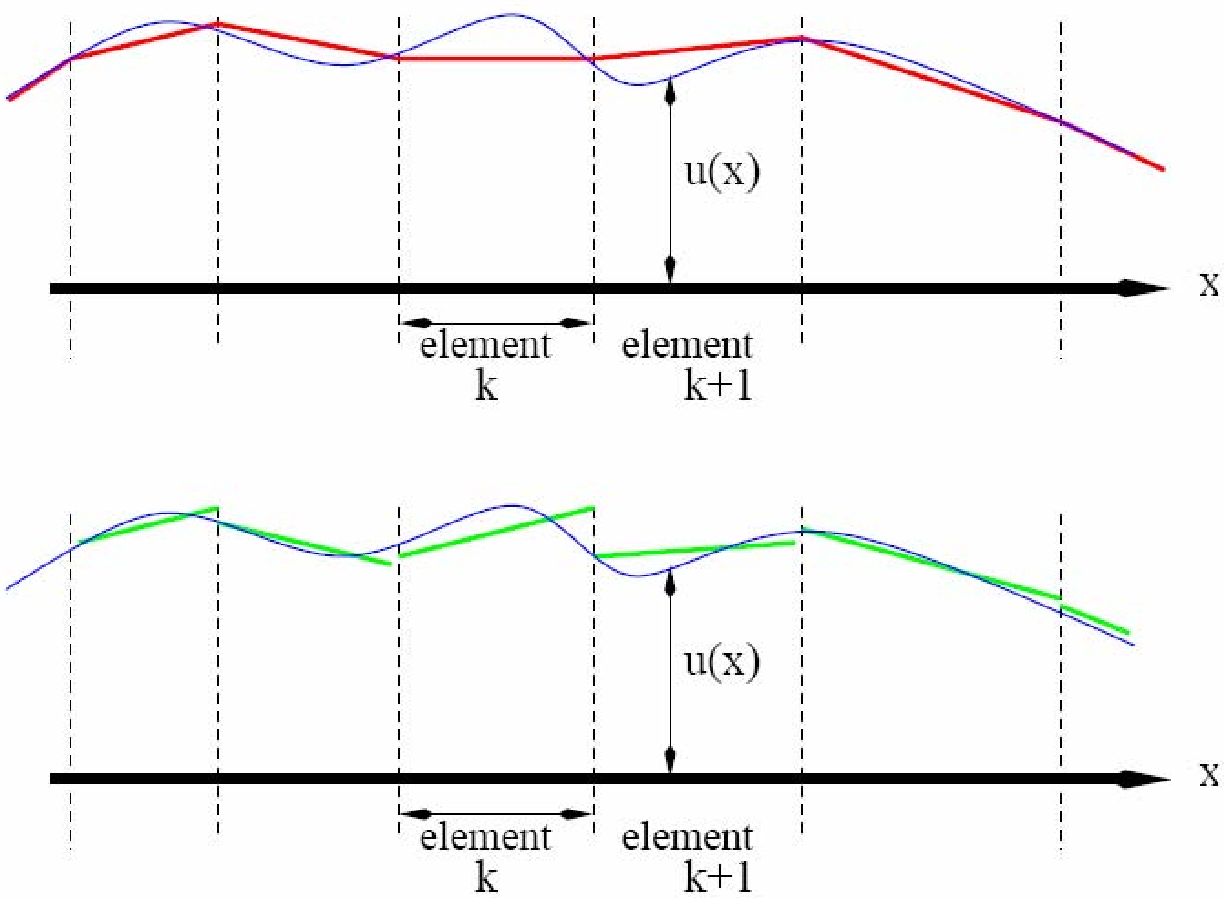


Figure 1.2: In a continuous Galerkin finite element method, the variable $u = u(x)$ is approximated globally in a (piecewise linear) continuous manner (top figure). In contrast, in a discontinuous Galerkin finite element method, the variable $u = u(x)$ is approximated globally in a discontinuous manner and locally in each element in a (piecewise linear) continuous way (bottom figure).

Reference:Introduction to (dis)continuous Galerkin finite element methods
by Onno Bokhove and Jaap J.W. van der Vegt

The second is that the DG methods are *locally conservative*.

This is a reflection of the fact that the method enforces the equation element-by-element *and* of the use of the numerical trace. In our simple setting, this property reads

$$\widehat{u}_h\Big|_{t^n}^{t^{n+1}} = \int_{I_n} f(s)\,u(s)\,ds,$$

and is obtained by simply taking *v* $\equiv$ 1 in the weak formulation (2). This a much valued property in computational fluid dynamics.

$$-\int_{I^n} u_h(s)\,\frac{d}{dt}v(s)\,ds + \widehat{u}_h\,v\Big|_{t^n}^{t^{n+1}} = \int_{I_n} f(s)\,u(s)\,v(s)\,ds,$$

The third property is the strong relation between the residuals of uh inside the intervals and its jumps across inter-interval boundaries.

To uncover it, let us integrate by parts in (2) to get

$$\int_{I^n} \frac{d}{dt} u_h(s)\, v(s)\, ds + (\widehat{u}_h - u_h)\, v\big|_{t^n}^{t^{n+1}} = \int_{I_n} f(s)\, u(s)\, v(s)\, ds,$$

or, equivalently,

$$\int_{I^n} R(s)\, v(s)\, ds = (u_h - \widehat{u}_h)\, v\big|_{t^n}^{t^{n+1}},$$

where R denotes the residual $\left(\frac{d}{dt} u_h - f\, u_h\right)$. If we now take $v = 1$ and use the definition of the numerical trace $\widehat{u}_h$, we obtain

$$\int_{I^n} R(s)\, ds = [\![u_h]\!](t^n).$$

In other words, the jump of u_h at t^n, $[\![u_h]\!](t^n)$, is nothing but the integral of the residual over the interval I^n.

we consider DG methods for the model elliptic problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain of $\mathbb{R}^d$. we rewrite elliptic model problem as

$$\boldsymbol{q} = \nabla u, \quad -\nabla \cdot \boldsymbol{q} = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

The DG methods.

A DG numerical method is obtained as follows.After discretizing the domain Ω, the approximate solution $(\boldsymbol{q}_h, u_h)$ on the element K is taken in the space $\mathcal{Q}(K) \times \mathcal{U}(K)$ and is determined by requiring that

$$\int_K \boldsymbol{q}_h \cdot \boldsymbol{v} \, dx = -\int_K u_h \, \nabla \cdot \boldsymbol{v} \, dx + \int_{\partial K} \widehat{u}_h \, \boldsymbol{v} \cdot \boldsymbol{n} \, ds,$$

$$\int_K \boldsymbol{q}_h \cdot \, \nabla w \, dx - \int_{\partial K} w \, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \, ds = \int_K f w \, dx,$$

for all $(\boldsymbol{v}, w) \in \mathcal{Q}(K) \times \mathcal{U}(K)$.

Note that now we have two numerical traces, namely, $\widehat{u}_h$ and $\widehat{q}_h$, that remain to be defined.

To do that, we begin by finding a stability result for the solution of the original equation. To do that, we multiply the first equation by $\boldsymbol{q}$ and integrate over Ω to get

$$\int_{\Omega} |\boldsymbol{q}|^2 \, dx - \int_{\Omega} \boldsymbol{q} \cdot \nabla u \, dx = 0.$$

Then, we multiply the second equation by *u* and integrate over Ω to obtain

$$-\int_{\Omega} \nabla \cdot \boldsymbol{q} \, u \, dx = \int_{\Omega} f \, u \, dx.$$

Adding these two equations, we get

$$\int_{\Omega} |\boldsymbol{q}|^2 \, dx = \int_{\Omega} f \, u \, dx.$$

This is the result we sought. Next, we mimic this procedure for the DG method.

We begin by taking $\boldsymbol{v} = \boldsymbol{q}_h$ in the first equation defining the DG method and adding on the elements K to get

$$\int_\Omega |\boldsymbol{q}_h|^2 \, dx - \sum_{K \in \mathcal{T}_h} \left(- \int_K u_h \nabla \cdot \boldsymbol{q}_h \, d\boldsymbol{x} + \int_{\partial K} \widehat{u}_h \, \boldsymbol{q}_h \cdot \boldsymbol{n} \, ds \right) = 0.$$

Next, we take $w = u_h$ in the second equation and add on the elements to obtain

$$\sum_{K \in \mathcal{T}_h} \left(\int_K \boldsymbol{q}_h \cdot \nabla u_h \, d\boldsymbol{x} - \int_{\partial K} u_h \, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \, ds \right) = \int_\Omega f u_h \, d\boldsymbol{x}.$$

Adding the two above equations, we find that

$$\int_\Omega |\boldsymbol{q}_h|^2 \, dx + \Theta_h = \int_\Omega f \, u \, dx,$$

where

$$\Theta_h = - \sum_{K \in \mathcal{T}_h} \left(- \int_K \nabla \cdot (u_h \, \boldsymbol{q}_h) \, d\boldsymbol{x} + \int_{\partial K} (\widehat{u}_h \, \boldsymbol{q}_h \cdot \boldsymbol{n} + u_h \, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}) \, ds \right).$$

It only remains to show that we can define consistent numerical traces $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$ that render Θ_h non-negative. Si

$$\Theta_h = -\sum_{K\in\mathcal{T}_h}\left(-\int_K \nabla\cdot(u_h\,\boldsymbol{q}_h)\,d\boldsymbol{x} + \int_{\partial K}(\widehat{u}_h\,\boldsymbol{q}_h\cdot\boldsymbol{n} + u_h\,\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n})\,ds\right).$$

It only remains to show that we can define consistent numerical traces $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$ that render Θ_h non-negative. Since,

$$\begin{aligned}
\Theta_h &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}(u_h\,\boldsymbol{q}_h\cdot\boldsymbol{n} - \widehat{u}_h\,\boldsymbol{q}_h\cdot\boldsymbol{n} - u_h\,\widehat{\boldsymbol{q}_h}\cdot\boldsymbol{n})\,ds \\
&= \sum_{e\in\mathcal{E}_h}\int_e [\![u_h\,\boldsymbol{q}_h - \widehat{u}_h\,\boldsymbol{q}_h - u_h\,\widehat{\boldsymbol{q}_h}]\!]\,ds \\
&= \sum_{e\in\mathcal{E}_{ih}}\int_e ([\![u_h\,\boldsymbol{q}_h]\!] - \widehat{u}_h\,[\![\boldsymbol{q}_h]\!] - [\![u_h]\!]\cdot\widehat{\boldsymbol{q}}_h)\,ds + \int_{\partial\Omega}(u_h\,\boldsymbol{q}_h - \widehat{u}_h\,\boldsymbol{q}_h\cdot\boldsymbol{n} - u_h\,\widehat{\boldsymbol{q}_h}\cdot\boldsymbol{n})\,ds \\
&= \sum_{e\in\mathcal{E}_{ih}}\int_e ((\{u_h\} - \widehat{u}_h)\,[\![\boldsymbol{q}_h]\!] + [\![u_h]\!]\cdot(\{\boldsymbol{q}\} - \widehat{\boldsymbol{q}_h}))\,ds + \int_{\partial\Omega}(u_h\,(\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} - \widehat{u}_h\,\boldsymbol{q}_h\cdot\boldsymbol{n})\,ds,
\end{aligned}$$

$$= \sum_{e \in \mathcal{E}_{ih}} \int_e \left(\left(\{u_h\} - \widehat{u}_h \right) [\![\boldsymbol{q}_h]\!] + [\![u_h]\!] \cdot \left(\{\boldsymbol{q}\} - \widehat{\boldsymbol{q}_h} \right) \right) ds + \int_{\partial\Omega} \left(u_h \left(\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h \right) \cdot \boldsymbol{n} - \widehat{u}_h \, \boldsymbol{q}_h \cdot \boldsymbol{n} \right) ds,$$

it is enough to take, inside the domain Ω,

$$\widehat{\boldsymbol{q}}_h = \{\boldsymbol{q}_h\} + C_{11} [\![u_h]\!] + \boldsymbol{C}_{12} [\![\boldsymbol{q}_h]\!], \qquad \widehat{u}_h = \{u_h\} - \boldsymbol{C}_{12} \cdot [\![u_h]\!] + C_{22} [\![\boldsymbol{q}_h]\!],$$

and on its boundary,

$$\widehat{\boldsymbol{q}_h} = \boldsymbol{q}_h - C_{11} \, u_h \, \boldsymbol{n}, \qquad \widehat{u}_h = 0,$$

to finally get

$$\Theta_h = \sum_{e \in \mathcal{E}_{ih}} \int_e \left(C_{22} \, [\![\boldsymbol{q}_h]\!]^2 + C_{11} \, [\![u_h]\!]^2 \right) ds + \int_{\partial\Omega} C_{11} \, u_h^2 \, ds \geq 0,$$

provided C_{11} and C_{22} are non-negative. Note how the boundary conditions are imposed weakly through the definition of the numerical traces. This completes the definition of the DG methods.

It is enough to take $\widehat{\overline{q_h}} = \{\overrightarrow{q_h}\} - C11 \times [[u_h]] + \overrightarrow{C12} \bullet [[\overrightarrow{q_h}]]$;

$$\hat{u} = \{u_h\} - \overrightarrow{C12} \bullet [[u_h]] - C22 \times [[\overrightarrow{q_h}]]$$ inside the domain Ω

and on its boundary

$\widehat{\overline{q_h}} = \overrightarrow{q_h} - C11 \times u_h \times \vec{n}$; $\hat{u} = 0$;

where $\{\vec{q}_h\} = \frac{1}{2} \times (\vec{q}^{\,+} + \vec{q}^{\,-})$

$$\{u_h\} = \frac{1}{2} \times (u_h^{\,+} + u_h^{\,-})$$

$$[[\vec{q}_h]] = \vec{q}^{\,+} \bullet \vec{n}^{\,+} + \vec{q}^{\,-} \bullet \vec{n}^{\,-}$$

$$[[u_h]] = u^+ \vec{n}^{\,+} + u^- \vec{n}^{\,-}$$

when solving $-\Delta u=f$ in Ω and $u=g(x,y)$ in $\partial\Omega$

we take $\widehat{\overline{q_h}} = \{\overline{q_h}\} - C11\times[[u_h]] + \overline{C12}\bullet[[\overline{q_h}]]$; $\hat{u} = \{u_h\} - \overline{C12}\bullet[[u_h]] - C22\times[[\overline{q_h}]]$;

and on its boundary

$\widehat{\overline{q_h}} = \overline{q_h} - C11\times u_h\times\vec{n} + C11\times g(x,y)\times\vec{n}$; $\hat{u} = g(x,y)$;

where $\{\vec{q}_h\} = \frac{1}{2}\times(\vec{q}^{+} + \vec{q}^{-})$

$$\{u_h\} = \frac{1}{2}\times(u_h^{+} + u_h^{-})$$

$$[[\vec{q}_h]] = \vec{q}^{+}\bullet\vec{n}^{+} + \vec{q}^{-}\bullet\vec{n}^{-}$$

$$[[u_h]] = u^{+}\vec{n}^{+} + u^{-}\vec{n}^{-}$$

Some properties.

(i) Let us show that to guarantee the existence and uniqueness of the approximate solution of the DG methods, the parameter *C*11 has to be greater than zero and the local spaces *U*(*K*) and *Q*(*K*) must satisfy the following *compatibility* condition:

$$u_h \in \mathcal{U}(K): \quad \int_K \nabla u_h \, v \, dx = 0 \quad \forall v \in \mathcal{Q}(K) \quad \text{then} \quad \nabla u_h = 0.$$

Indeed, the approximate solution is well defined if and only if, the only approximate solution to the problem with *f* = 0 is the trivial solution.

In that case, our stability identity (page16) gives

$$\int_\Omega |\boldsymbol{q}_h|^2 \, dx + \sum_{e \in \mathcal{E}_{ih}} \int_e \left(C_{22} \, [\![\boldsymbol{q}_h]\!]^2 + C_{11} \, [\![u_h]\!]^2 \right) \, ds + \int_{\partial\Omega} C_{11} \, u_h^2 \, ds = 0,$$

$$\int_{\Omega} |q_h|^2 \, dx + \sum_{e \in \mathcal{E}_{ih}} \int_e \left(C_{22} \, [\![q_h]\!]^2 + C_{11} \, [\![u_h]\!]^2 \right) \, ds + \int_{\partial\Omega} C_{11} \, u_h^2 \, ds = 0,$$

which implies that $\boldsymbol{q}_h = 0$, $[\![u_h]\!] = 0$ on $\mathcal{E}_{ih}$, and $u_h = 0$ on $\partial\Omega$, provided that $C_{11} > 0$. We can now rewrite the first equation defining the method as follows:

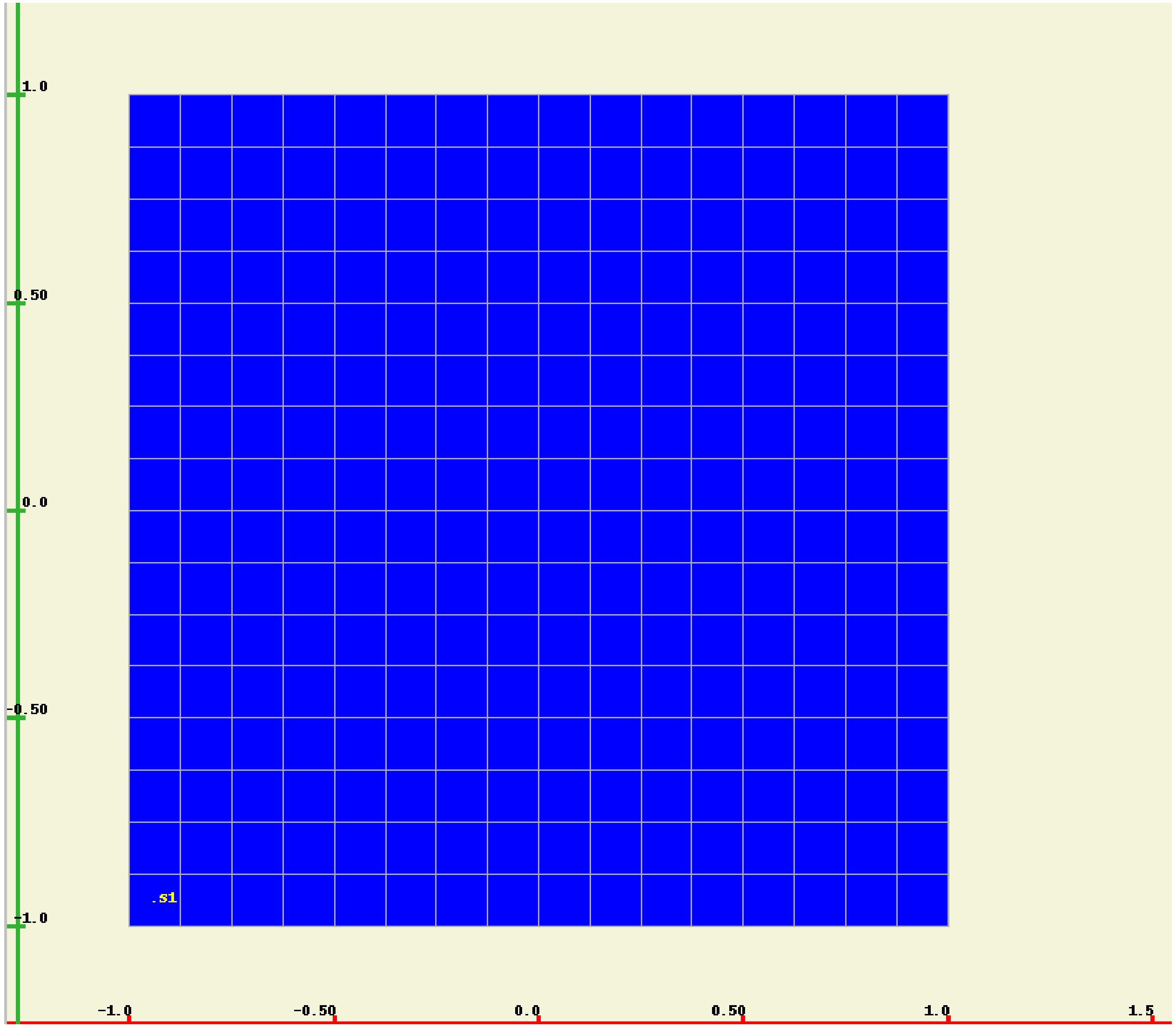
$$\int_K \nabla u_h \, \boldsymbol{v} \, dx = 0, \quad \forall \, \boldsymbol{v} \in \mathcal{Q}_h,$$

which, by the compatibility condition, implies that $\nabla u_h = \mathbf{0}$. Hence $u_h = 0$, as wanted.

(ii) When all the local spaces contain the polynomials of degree *k*, the orders of convergence of the *L*2-norms of the errors in ***q*** and *u* are *k* and *k* + 1, respectively.when *C*11 is of order *O*(*h*^ −1).

Example:

Domain: [-1,1]*[-1,1]



c11=1/h,c12=(0,0),c22=0,

u=x^2+y^2+x^2*y+x*y^2+x^2*y^2+x+y+x*y+1;

n=2 (n:degree of legendre polynomial)

nit	u_x error(2-norm)	u_y error(2-norm)	u error(2-norm)
1	1.798717e-015	2.302556e-015	9.280619e-016
2	4.484038e-015	4.137567e-015	1.314049e-015
4	5.829104e-015	5.534534e-015	1.601673e-015
8	9.283783e-015	8.986503e-015	1.986222e-015
16	1.280688e-014	1.247376e-014	1.699789e-015
32	2.081884e-014	2.007377e-014	1.988631e-015

u=x^2+y^2+x^2*y+x*y^2+x^2*y^2+x+y+x*y+1;

n=1;
c11=1;c12=(0,0);c22=1;

nit	u error(2-norm)
1	1.324597e+000
2	3.251126e-001
4	7.934654e-002
8	1.969471e-002
16	4.907614e-003
32	1.224563e-003

nit	u Order
1	2.0158
2	2.0158
3	2.0158
4	2.0158
5	2.0158

(iii)DG methods are in fact mixed finite element methods. To see this, let us begin by noting that the DG approximate solution (***q**h, uh*) can be also be characterized as the solution of

$$a(q_h, v) + b(u_h, v) = 0,$$

$$-b(w, q_w) + c(u_h, w) = F(w),$$

for all $(v, w) \in \mathcal{Q}_h \times \mathcal{U}_h$ where

$$\mathcal{Q}_h = \{v : v \in \mathcal{Q}(K)\ \forall K \in \mathcal{T}_h\}, \qquad \mathcal{U}_h = \{w : w \in \mathcal{U}(K)\ \forall K \in \mathcal{T}_h\},$$

$$a(\boldsymbol{q}_h, \boldsymbol{v}) + b(u_h, \boldsymbol{v}) = 0,$$

$$-b(w, \boldsymbol{q}_w) + c(u_h, w) = F(w),$$

for all $(\boldsymbol{v}, w) \in \mathcal{Q}_h \times \mathcal{U}_h$ where

$$\mathcal{Q}_h = \{\boldsymbol{v} : \boldsymbol{v} \in \mathcal{Q}(K) \ \forall\ K \in \mathcal{T}_h\}, \qquad \mathcal{U}_h = \{w : w \in \mathcal{U}(K) \ \forall\ K \in \mathcal{T}_h\},$$

and

$$a(\boldsymbol{q}, \boldsymbol{r}) := \int_\Omega \boldsymbol{q} \cdot \boldsymbol{r}\, dx + \int_{\mathcal{E}_i} C_{22} [\![\boldsymbol{q}]\!] [\![\boldsymbol{r}]\!]\, ds,$$

$$b(u, \boldsymbol{r}) := \sum_{K \in \mathcal{T}} \int_K u \nabla \cdot \boldsymbol{r}\, dx - \int_{\mathcal{E}_i} (\{u\} + \boldsymbol{C}_{12} \cdot [\![u]\!])\ [\![\boldsymbol{r}]\!]\, ds,$$

$$c(u, v) := \int_{\mathcal{E}_{ih}} C_{11}\, [\![u]\!] \cdot [\![v]\!]\, ds + \int_{\partial\Omega} C_{11}\, uv\, ds,$$

$$F(\boldsymbol{r}) := \int_\Omega f v\, dx.$$

As a consequence, the corresponding matrix equation has the form

$$\begin{pmatrix} \mathbb{A} & -\mathbb{B}^t \\ \mathbb{B} & \mathbb{C} \end{pmatrix} \begin{pmatrix} \mathbb{Q} \\ \mathbb{U} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{F} \end{pmatrix},$$

which is typical of stabilized mixed finite element methods.

those methods are not well defined unless the ‘stabilizing’ form $c(\cdot, \cdot)$, usually associated with residuals, is introduced.

For DG methods, the ‘stabilizing’ form $c(\cdot, \cdot)$ solely depends on the parameter $C11$ and the jumps across elements of the functions in Uh.

This is why we could think that this form stabilizes the method by *penalizing the jumps*, $C11$ being the *penalization* parameter;

(iv) The methods we have presented are locally conservative.

As in the hyperbolic case, this is a reflection of the form of the weak formulation and the fact that the definition of the numerical traces on the face *e* does not depend on what side of it we are.

More general DG methods define the approximate solution by requiring that

$$\int_K \boldsymbol{q}_h \cdot \boldsymbol{r}\, dx = -\int_K u_h \nabla \cdot \boldsymbol{r}\, dx + \int_{\partial K} \widehat{u}_{h,K}\, \boldsymbol{r} \cdot \boldsymbol{n}_K\, ds,$$

$$\int_K \boldsymbol{q}_h \cdot \nabla v\, dx = \int_K f v\, dx + \int_{\partial K} v\, \widehat{\boldsymbol{q}}_{h,K} \cdot \boldsymbol{n}_K\, ds,$$

for all (***r***, *v*) $\in$ ***Q***(*K*) $\times$ *U*(*K*).
In this general formulation, the numerical traces *uh,K* and ***q****h,K can* have definitions that *might* depend on what side of the element boundaries we are.
Hence they are not locally conservative. This is the case for the numerical fluxes in *u* of the last four schemes in Table 2.

Table 2 Some DG methods and their numerical fluxes.

Method	$\widehat{\boldsymbol{q}}_{e,K}$	$\widehat{u}_{h,K}$
Bassi–Rebay [10]	$\{\boldsymbol{q}_h\}$	$\{u_h\}$
LDG [38]	$\{\boldsymbol{q}_h\} + C_{11}\,[\![u_h]\!] - \boldsymbol{C}_{12}[\![\boldsymbol{q}_h]\!]$	$\{u_h\} + \boldsymbol{C}_{12}\cdot[\![u_h]\!]$
DG [22]	$\{\boldsymbol{q}_h\} + C_{11}\,[\![u_h]\!] - \boldsymbol{C}_{12}[\![\boldsymbol{q}_h]\!]$	$\{u_h\} + \boldsymbol{C}_{12}\cdot[\![u_h]\!] + C_{22}[\![\boldsymbol{q}_h]\!]$
Brezzi et al. [20]	$\{\boldsymbol{q}_h\} - \alpha^r\,([\![u_h]\!])$	$\{u_h\}$
IP [45]	$\{\nabla u_h\} + C_{11}\,[\![u_h]\!]$	$\{u_h\}$
Bassi–Rebay [12]	$\{\nabla u_h\} - \alpha^r\,([\![u_h]\!])$	$\{u_h\}$
Baumann–Oden [15]	$\{\nabla u_h\}$	$\{u_h\} - \boldsymbol{n}_K\cdot[\![u_h]\!]$
NIPG [70]	$\{\nabla u_h\} + C_{11}[\![u_h]\!]$	$\{u_h\} - \boldsymbol{n}_K\cdot[\![u_h]\!]$
Babuška–Zlámal [7]	$C_{11}[\![u_h]\!]$	$u_h\vert_K$
Brezzi et al. [20]	$-\alpha^r\,([\![u_h]\!])$	$u_h\vert_K$

the function α^r([*uh*]) is a special stabilization term introduced by Bassi and Rebay [12] and later studied by Brezzi et al. [20]; its stabilization properties are equivalent to the one originally presented.

Consider the Laplace eigenproblem $-\Delta u=\lambda u$ in Ω and $u=0$ in $\partial\Omega$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d$, $d=2,3$

Solving the eigenproblem with LDG method

λ is the eigenvalue of the matrix $C-BA^{-1}(-B^T)$

$$\begin{pmatrix} \mathbb{A} & -\mathbb{B}^t \\ \mathbb{B} & \mathbb{C} \end{pmatrix} \begin{pmatrix} \mathbb{Q} \\ \mathbb{U} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{F} \end{pmatrix},$$

THANK YOU

Reference

Bernardo Cockburn,Discontinuous Galerkin methods

Paul Castillo,Performance of discontinuous Galerkin methods for elliptic pdes

P. Antonietti, A. Buffa and I. Perugia ,Discontinuous Galerkin Approximation of the Laplace eigenproblem

Onno Bokhove and Jaap J.W. van der Vegt,Introduction to (dis)continuous Galerkin finite element methods