High-Order SBP Implicit Difference Operator

§ One-dimensional scalar coefficient system

Consider the advection scalar system:

$$q_t + Aq_x = 0, x \in I = [0,1], t \geq 0, \text{ A : 2 by 2 symmetric matrix}$$
$$q = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$$

Define $E(t) = \int_I \langle q, q \rangle dx$,

$$\frac{1}{2}\frac{d}{dt}E(t) = \frac{1}{2}\int_0^1 (\frac{\partial q^T}{\partial t} q + q^T \frac{\partial q}{\partial t})dx = \int_0^1 q^T \frac{\partial q}{\partial t} dx = -\int_0^1 q^T (Aq_x) dx = -q^T Aq|_0^1$$

Since A is symmetric, A is diagonalizable with othogonal eigenvectors, and then $A = R\Lambda R^{-1}$, where $\Lambda = diag\,(\lambda_1, \lambda_2)$, and $R = \begin{pmatrix} R_1 & R_2 \end{pmatrix}$ is the matrix of eigenvectors corresponding to Λ, in which, orthogonal R_i's are available. Notice that, $A = A^+ + A^-$, where $A^+ = R\Lambda^+ R$ and $A^- = R\Lambda^- R$.

$$\Lambda^+ = diag\,(Max\,(\lambda_1, 0)\,, Max\,(\lambda_2, 0))\,,$$

$$\Lambda^- = diag\,(min\,(\lambda_1, 0)\,, min\,(\lambda_2, 0))$$

Hence,

$$\begin{aligned} \frac{1}{2}\frac{dE(t)}{dt} &= -q^T(A^+ + A^-)q|_0^1 = -q^T A^+ q|_0^1 - q^T A^- q|_0^1 \\ &\leq q^T A^+ q|_{x=0} - q^T A^- q|_{x=1} \end{aligned}$$

$$\Rightarrow \frac{dE(t)}{dt} \leq 0 \text{ provided that } A^+ q|_{x=0} \leq 0 \text{ and } A^- q|_{x=1} \geq 0$$

Mesh $I = [0,1]$ into uniform grids $x = (x_0, x_1, \ldots, x_L)$, with $h_x = \frac{1}{L} = x_i - x_{i-1}\,(1 \leq i \leq L)$, the semi-discrete scheme with SAT can be written as:

$$\frac{d}{dt}V = -(D \otimes A)V + r,\ D = h_x^{-1} P^{-1} Q,\ r = r_1 + r_2$$

where

$$r_1 = h_x^{-1} \tau^0 q_{00} (P^{-1} e_0) \otimes [\ \frac{1}{2}(A + |A|)(V_0(t) - V_0^{BC}(t)\ )\ ]$$
$$r_2 = h_x^{-1} \tau^L q_{LL} (P^{-1} e_L) \otimes [\ \frac{1}{2}(A - |A|)(V_L(t) - V_L^{BC}(t)\ )\ ]$$

In which, $|A| = R|\Lambda|R^{-1} = R \cdot diag\,(|\lambda_1|, |\lambda_2|) \cdot R^{-1}$, $V \in Vec_{(L+1)\otimes 2}$, and $e_i = (\delta_{0i}, \delta_{1i}, \delta_{2i}, \ldots, \delta_{Li})^T$,$V_0^{BC}$ and V_L^{BC} respectively denote the boundary conditions for $x = x_0$ and $x = x_L$.

$$V = \sum_{i=0}^{L} e_i \otimes V_i = \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_L \end{pmatrix} = \begin{pmatrix} (u_0, v_0)^T \\ (u_1, v_1)^T \\ \vdots \\ (u_L, v_L)^T \end{pmatrix}.$$

(**Remark**.) Q and $P \in M_{L+1}$, and satisfy the **SBP** properties.

SBP1: The matrix P is symmetric and positive definite.

SBP2: The matrix Q is nearly skew-symmetric, that is,

$$\frac{Q+Q^T}{2} = diag\left(q_{00}, 0, \ldots, 0, q_{LL}\right), \; q_{00} < 0, \; q_{LL} = -q_{00} > 0$$

We now derive the energy estimation for the semi-discrete scheme.

Define $E\left(t\right) = \|V\|^2_{h_x,P} = \left(V, V\right)_{h_x,P}$, then

$$\begin{aligned}\frac{1}{2}\frac{dE(t)}{dt} &= (V, \frac{d}{dt}V)_{h_x,P} = (V, -(D\otimes A)V + r)_{h_x,P} \\ &= -(V, (h_x^{-1}P^{-1}Q\otimes A)V)_{h_x,P} + (V, r)_{h_x,P} \\ &= -(V, (Q\otimes A)V) + (V, r_1)_{h_x,P} + (V, r_2)_{h_x,P} \; .\end{aligned}$$

Consider $(V, (Q\otimes A)\,V)$,

$$(V, (Q\otimes A)V) = \sum_{i,i'=0}^{L} (e_i^T \otimes V_i^T)(Q\otimes A)(e_{i'}\otimes V_{i'}) = \sum_{i,i'=0}^{L} (e_i^T Q e_{i'})(V_i^T A V_{i'})$$

In which, $Q = Q^S + Q^A$, where $Q^S = \frac{Q+Q^T}{2} = diag\left(q_{00}, 0, \ldots, 0, q_{LL}\right)$ and $Q^A = \frac{Q-Q^T}{2}$ are respectively the symmetric part and anti-symmetric part of matrix Q. Thus,

$$\begin{aligned}(V, (Q\otimes A)V) &= \sum_{i,i'=0}^{L} (e_i^T (Q^S + Q^A) e_{i'})(V_i^T A V_{i'}) = \sum_{i,i'=0}^{L} (e_i^T Q^S e_{i'})(V_i^T A V_{i'}) + \sum_{i,i'=0}^{L} (e_i^T Q^A e_{i'})(V_i^T A V_{i'}) \\ &= \sum_{i,i'=0}^{L} (e_i^T Q^S e_{i'})(V_i^T A V_{i'}) \\ &= \sum_{i,i'=0}^{L} (e_i^T \cdot diag(q_{00}, 0, \ldots, 0, q_{LL}) \cdot e_{i'})(V_i^T (A^+ + A^-) V_{i'}) \\ &= q_{00} V_0^T A^+ V_0 + q_{LL} V_L^T A^+ V_L + q_{00} V_0^T A^- V_0 + q_{LL} V_L^T A^- V_L \\ &= q_{00}(V_0, V_0)_{A^+} + q_{LL}(V_L, V_L)_{A^+} + q_{00}(V_0, V_0)_{A^-} + q_{LL}(V_L, V_L)_{A^-} .\end{aligned}$$

(**Lemma1**.) $|A| = |A^+| + |A^-| = A^+ - A^-$
(**Lemma2**.) $A^+ \geq 0$, and $A^- \leq 0$.

Consider $(V, r_1)_{h_x,P}$, let $V_0^{BC}\left(t\right) = 0$, then

$$\begin{aligned}(V, r_1)_{h_x,P} &= (V, h_x^{-1}\tau^0 q_{00}(P^{-1}e_0)\otimes[\frac{1}{2}(A + |A|)V_0])_{h_x,P} = \tau^0 q_{00}(V, e_0\otimes(A^+V_0)) \\ &= q_{00}\tau^0 \sum_{i=0}^{L}(e_i^T\otimes V_i^T)(e_0\otimes(A^+V_0)) = q_{00}\tau^0\sum_{i=0}^{L}(e_i^T e_0)(V_i^T A^+ V_0) \\ &= q_{00}\tau^0\sum_{i=0}^{L}\delta_{i0}(V_i^T A^+ V_0) = q_{00}\tau^0 V_0^T A^+ V_0 = q_{00}\tau^0(V_0, V_0)_{A^+}\end{aligned}$$

It is similar to $(V, r_2)_{h_x, P}$,

$$(V, r_2)_{h_x, P} = q_{LL} \tau^L V_L^T A^- V_L = q_{LL} \tau^L (V_L, V_L)_{A^-} \ ,$$

and hence

$$(V, r)_{h_x, P} = q_{00} \tau^0 \, (V_0, V_0)_{A^+} + q_{LL} \tau^L \, (V_L, V_L)_{A^-} \ \ .$$

Thus,

$$\begin{aligned}
\frac{1}{2} \frac{dE(t)}{dt} &= (-q_{00}(V_0, V_0)_{A^+} - q_{LL}(V_L, V_L)_{A^+} - q_{00}(V_0, V_0)_{A^-} - q_{LL}(V_L, V_L)_{A^+}) \\
&\quad + (q_{00} \tau^0 (V_0, V_0)_{A^+} + q_{LL} \tau^L (V_L, V_L)_{A^-}) \\
&= -q_{00}(1 - \tau^0)(V_0, V_0)_{A^+} - q_{LL}(V_L, V_L)_{A^+} - q_{00}(V_0, V_0)_{A^-} - q_{LL}(1 - \tau^L)(V_L, V_L)_{A^-}) \\
&\leq -q_{00}(1 - \tau^0)(V_0, V_0)_{A^+} - q_{LL}(1 - \tau^L)(V_L, V_L)_{A^-}
\end{aligned}$$

$\frac{d}{dt} E\,(t) \leq 0$ provided that $\tau^0 \geq 1$ and $\tau^L \geq 1$.

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