1. $\begin{cases} u_t = d\ u_{xx} + v \\ v_t = d\ v_{xx} + u \end{cases}$

I.C.$\begin{cases} u(x,0) = e^x \\ v(x,0) = 0 \end{cases}$ B.C.$\begin{cases} u(0,t) = e^{dt}\ \cosh t, & u(\pi,t) = e^{\pi+dt}\ \cosh t \\ v(0,t) = e^{dt}\ \sinh t, & v(\pi,t) = e^{\pi+dt}\ \sinh t \end{cases}$

Then, the exact solution is $\begin{cases} u(x,t) = e^{x+dt}\ \cosh t \\ v(x,t) = e^{x+dt}\ \cosh t \end{cases}$.

We take $x_i = ih,\ t_n = nk,$ where $h = \frac{1}{m+1},\ k = \frac{1}{n+1}$.

By Crank-Nicolson method,

$$\Rightarrow \begin{cases} \frac{U_i^{n+1} - U_i^n}{k} = \frac{d}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) + V_i^n \\ \frac{V_i^{n+1} - V_i^n}{k} = \frac{d}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) + U_i^n \end{cases}$$

$$\Rightarrow \begin{cases} U_i^{n+1} = U_i^n + \frac{dk}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) + kV_i^n \\ V_i^{n+1} = V_i^n + \frac{dk}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) + kU_i^n \end{cases}$$

Let $\gamma = \frac{k}{2h^2}$.

$$\Rightarrow \begin{cases} -\gamma d\ U_{i-1}^{n+1} + (1+2\gamma d)\ U_i^{n+1} - \gamma d\ U_{i+1}^{n+1} = \gamma d\ U_{i-1}^n + (1-2\gamma d)\ U_i^n + \gamma d\ U_{i+1}^n + kV_i^n \\ -\gamma d\ V_{i-1}^{n+1} + (1+2\gamma d)\ V_i^{n+1} - \gamma d\ V_{i+1}^{n+1} = \gamma d\ V_{i-1}^n + (1-2\gamma d)\ V_i^n + \gamma d\ V_{i+1}^n + kU_i^n \end{cases}$$

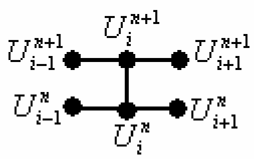
$$\begin{bmatrix} 1+2\gamma d & -\gamma d & 0 & & & 0 \\ -\gamma d & 1+2\gamma d & -\gamma d & & & \\ & & & \ddots & & \\ & & & -\gamma d & 1+2\gamma d & -\gamma d \\ 0 & & & 0 & -\gamma d & 1+2\gamma d \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \\ U_m^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma d\ (U_0^{n+1} + U_0^n) + (1-2\gamma d)\ U_1^n + \gamma d\ U_2^n + kV_1^n \\ \gamma d\ U_1^n + (1-2\gamma d)\ U_2^n + \gamma d\ U_3^n + kV_2^n \\ \vdots \\ \gamma d\ U_{m-2}^n + (1-2\gamma d)\ U_{m-1}^n + \gamma d\ U_m^n + kV_{m-1}^n \\ \gamma d\ U_{m-1}^n + (1-2\gamma d)\ U_m^n + \gamma d\ (U_{m+1}^{n+1} + U_{m+1}^n) + kV_m^n \end{bmatrix}$$

, and $$\begin{bmatrix} 1+2\gamma d & -\gamma d & 0 & & & 0 \\ -\gamma d & 1+2\gamma d & -\gamma d & & & \\ & & & \ddots & & \\ & & & -\gamma d & 1+2\gamma d & -\gamma d \\ 0 & & & 0 & -\gamma d & 1+2\gamma d \end{bmatrix} \begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ V_{m-1}^{n+1} \\ V_m^{n+1} \end{bmatrix}$$

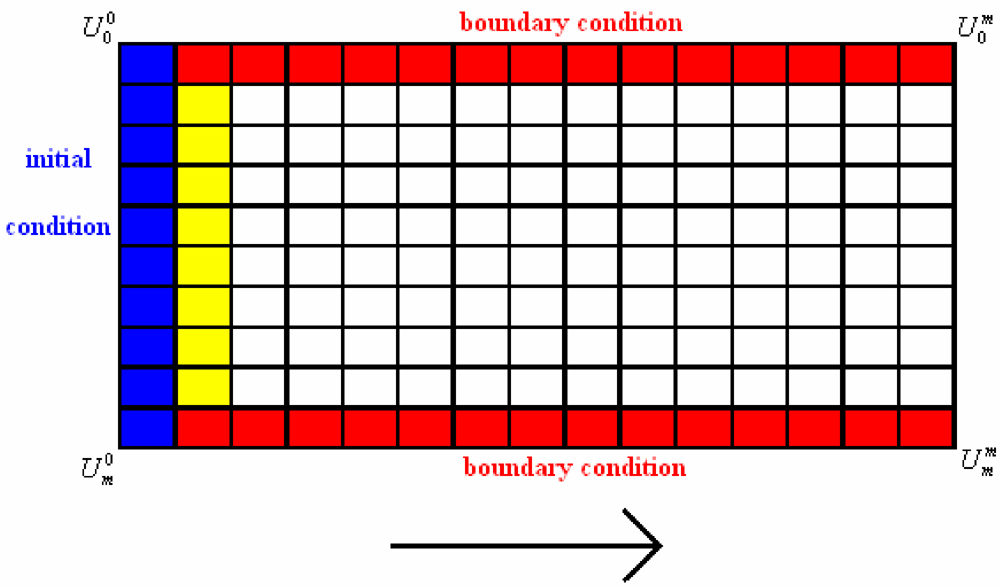
$$= \begin{bmatrix} \gamma d\ (V_0^{n+1} + V_0^n) + (1-2\gamma d)\ V_1^n + \gamma d\ V_2^n + kU_1^n \\ \gamma d\ V_1^n + (1-2\gamma d)\ V_2^n + \gamma d\ V_3^n + kU_2^n \\ \vdots \\ \gamma d\ V_{m-2}^n + (1-2\gamma d)\ V_{m-1}^n + \gamma d\ V_m^n + kU_{m-1}^n \\ \gamma d\ V_{m-1}^n + (1-2\gamma d)\ V_m^n + \gamma d\ (V_{m+1}^{n+1} + V_{m+1}^n) + kU_m^n \end{bmatrix}$$

在這個題目當中，我們使用了 Crank-Nicolson method 的隱式算法。應此可以得知我們所需要的點如下圖(一)，而 stability 的條件也就沒什麼限制。



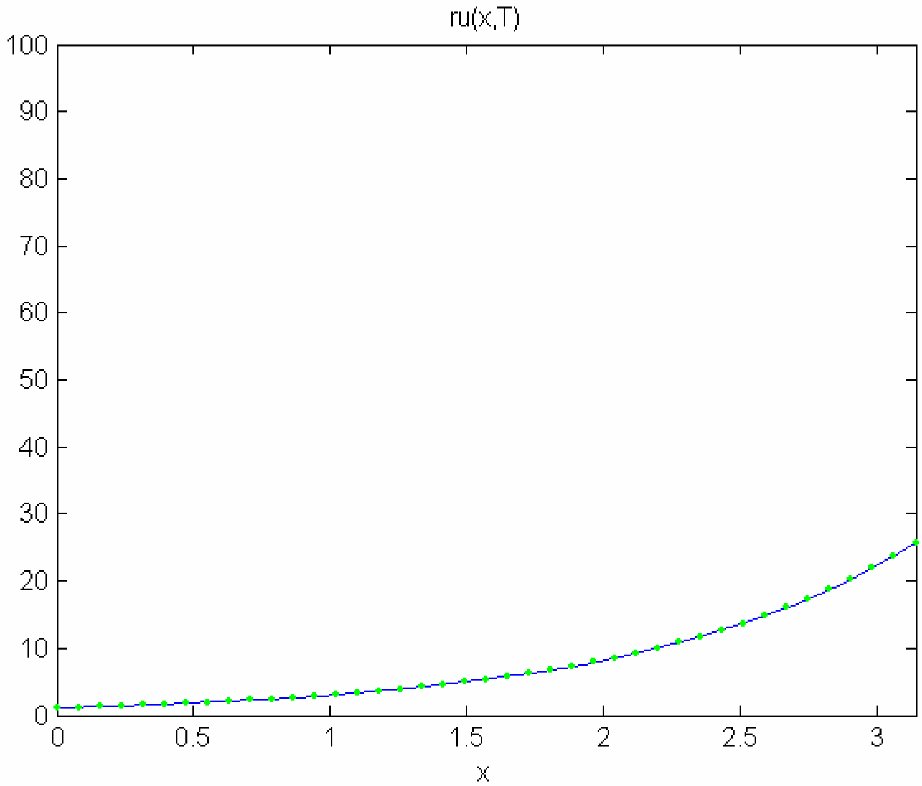
圖(一)

我們給 initial condition 及 boundary condition，可以得到的圖形為圖(二)，在這個圖形之中，我們給定紅色為 boundary condition；藍色為 initial condition；黃色為我們一次所解出來的解。而我們所解順序為從時間 $t_1 \to t_n$，解出在同一時間下的 U^t。而 V 的狀況也是在類似的條件下，因此我們就不贅述了。

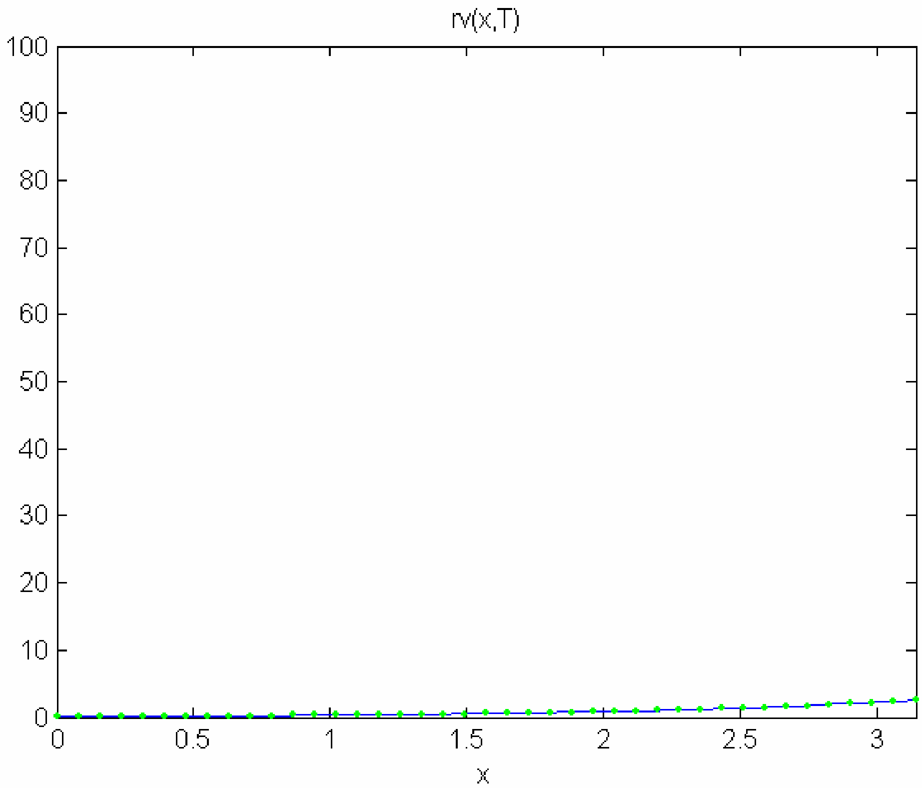


圖(二)

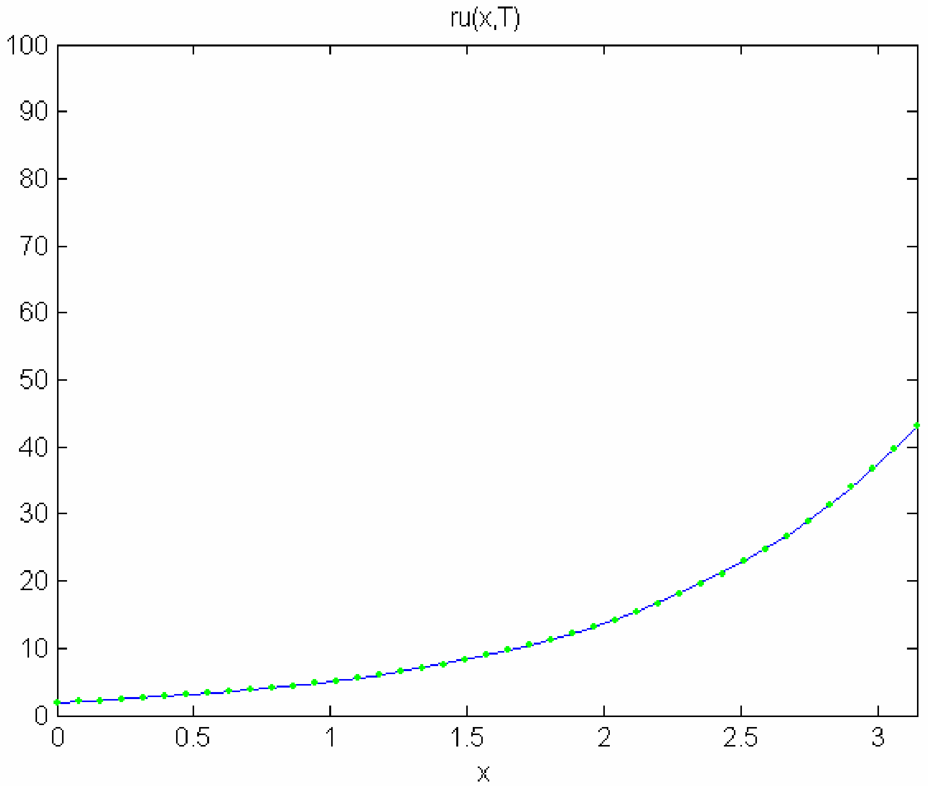
接下來我們以 $M = 39$ 來跑整個數值解，並且展示出 $t = 0.1, t = 0.5, t = 1$ 等這三個時間點的圖形。



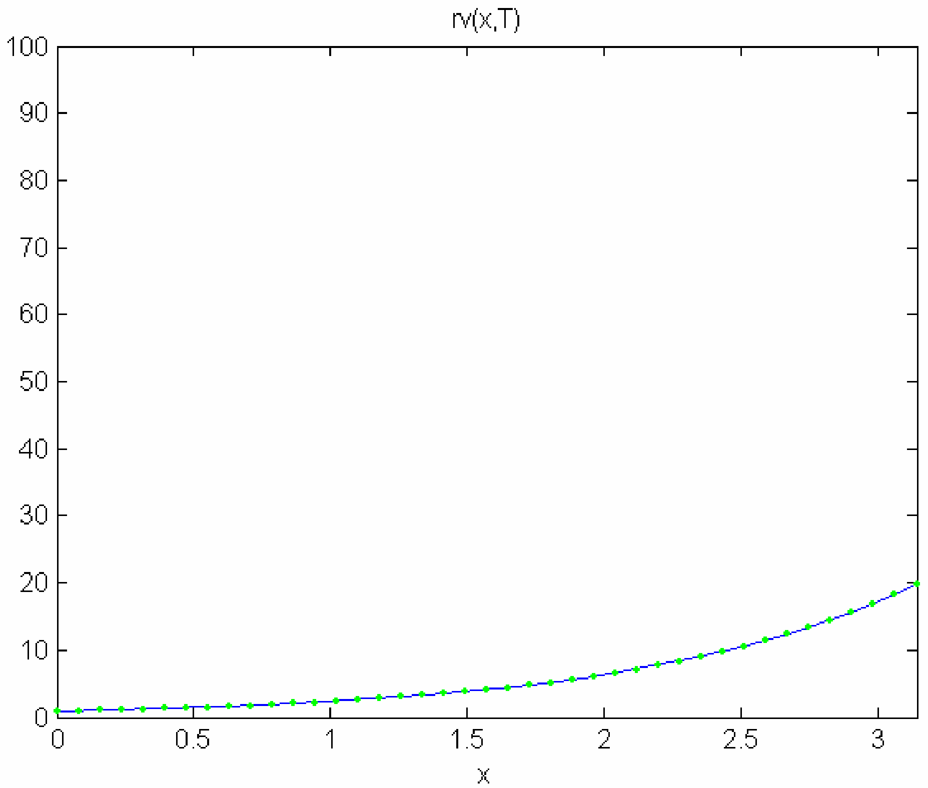
函數u在時間為$t=0.1$的圖形



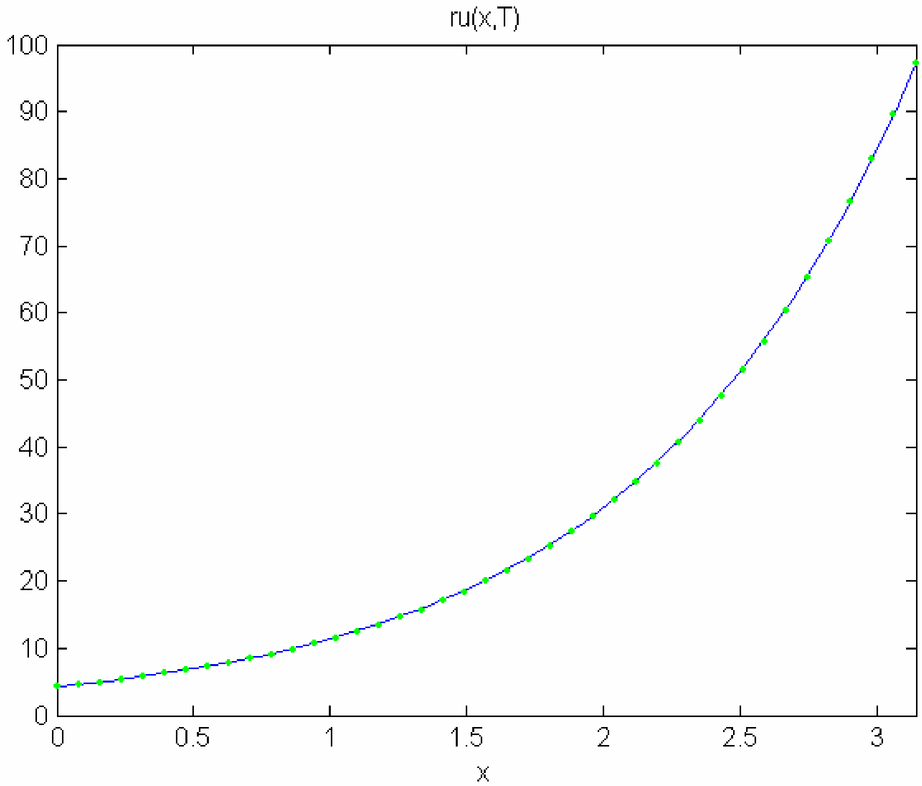
函數v在時間為$t=0.1$的圖形



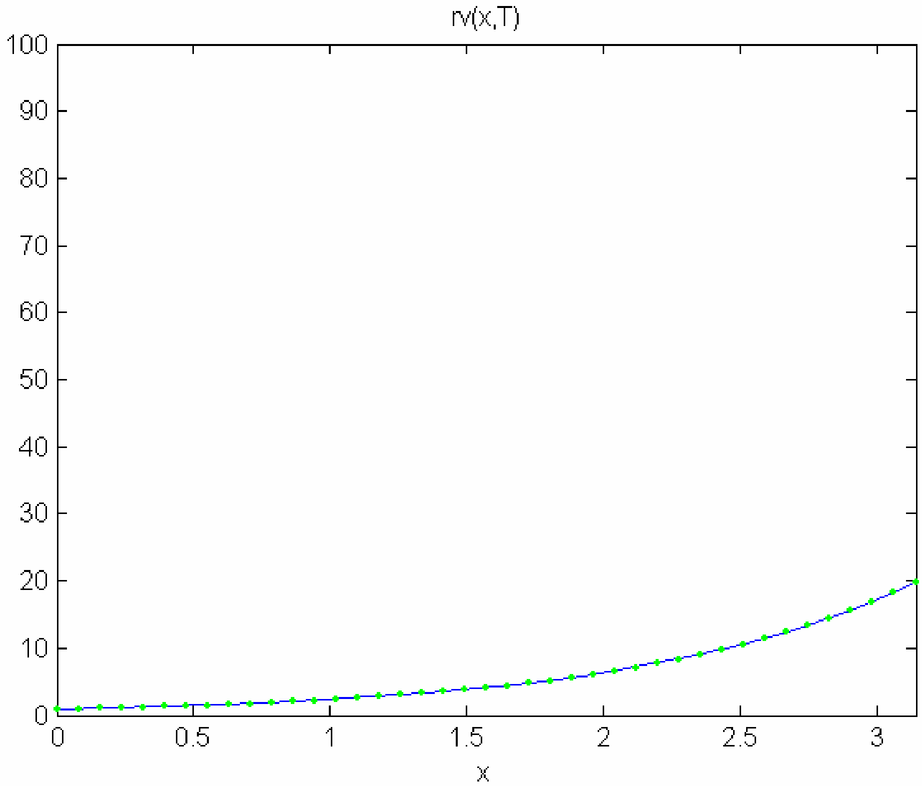
函數u在時間為$t=0.5$的圖形



函數v在時間為$t=0.5$的圖形



函數u在時間為$t=1$的圖形



函數v在時間為$t=1$的圖形

2. $\begin{cases} u_t = d\ u_{xx} - a\ u + v \\ v_t = d\ v_{xx} - bv \end{cases}$

$\begin{cases} u_x(0,t) = 0 \\ v_x(0,t) = 0 \end{cases}$ I.C. $\begin{cases} u(x,0) = 2\cos x \\ v(x,0) = (a-b)\cos x \end{cases}$ B.C. $\begin{cases} u(\frac{\pi}{2},t) = 0 \\ v(\frac{\pi}{2},t) = 0 \end{cases}$

Then, the exact solution is $\begin{cases} u(x,t) = (e^{-(a+d)t} + e^{-(b+d)t})\ \cos x \\ v(x,t) = (a-b)\ e^{-(b+d)t}\ \cos x \end{cases}$.

We take $x_i = ih,\ t_n = nk,$ where $h = \frac{1}{m+1},\ k = \frac{1}{n+1}$.

By Crank-Nicolson method,

$$\Rightarrow \begin{cases} \frac{U_i^{n+1} - U_i^n}{k} = \frac{d}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) - aU_i^n + V_i^n \\ \frac{V_i^{n+1} - V_i^n}{k} = \frac{d}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) - bV_i^n \end{cases}$$

$$\Rightarrow \begin{cases} U_i^{n+1} = U_i^n + \frac{dk}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) - akU_i^n + kV_i^n \\ V_i^{n+1} = V_i^n + \frac{dk}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) - bkV_i^n \end{cases}$$

Let $\gamma = \frac{k}{2h^2}$.

$$\Rightarrow \begin{cases} -\gamma d\ U_{i-1}^{n+1} + (1+2\gamma d)\ U_i^{n+1} - \gamma d\ U_{i+1}^{n+1} = \gamma d\ U_{i-1}^n + (1 - 2\gamma d - ak)\ U_i^n + \gamma d\ U_{i+1}^n + kV_i^n \\ -\gamma d\ V_{i-1}^{n+1} + (1+2\gamma d)\ V_i^{n+1} - \gamma d\ V_{i+1}^{n+1} = \gamma d\ V_{i-1}^n + (1 - 2\gamma d - bk)\ V_i^n + \gamma d\ V_{i+1}^n \end{cases}$$

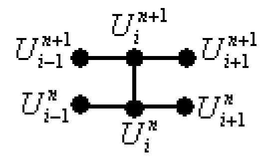
$$\begin{bmatrix} 1+2\gamma d & -2\gamma d & 0 & & & 0 \\ -\gamma d & 1+2\gamma d & -\gamma d & & & \\ & & & \ddots & & \\ & & & -\gamma d & 1+2\gamma d & -\gamma d \\ 0 & & & 0 & -\gamma d & 1+2\gamma d \end{bmatrix} \begin{bmatrix} V_0^{n+1} \\ V_1^{n+1} \\ \vdots \\ V_{m-1}^{n+1} \\ V_m^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} (1 - 2\gamma d - bk)\ V_1^n + 2\gamma d\ V_1^n \\ \gamma d\ V_0^n + (1 - 2\gamma d - bk)\ V_1^n + \gamma d\ V_2^n \\ \vdots \\ \gamma d\ V_{m-2}^n + (1 - 2\gamma d - bk)\ V_{m-1}^n + \gamma d\ V_m^n \\ \gamma d\ V_{m-1}^n + (1 - 2\gamma d - bk)\ V_m^n + \gamma d\ (V_{m+1}^{n+1} + V_{m+1}^n) \end{bmatrix}$$

, and $$\begin{bmatrix} 1+2\gamma d & -2\gamma d & 0 & & & 0 \\ -\gamma d & 1+2\gamma d & -\gamma d & & & \\ & & & \ddots & & \\ & & & -\gamma d & 1+2\gamma d & -\gamma d \\ 0 & & & 0 & -\gamma d & 1+2\gamma d \end{bmatrix} \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \\ U_m^{n+1} \end{bmatrix}$$

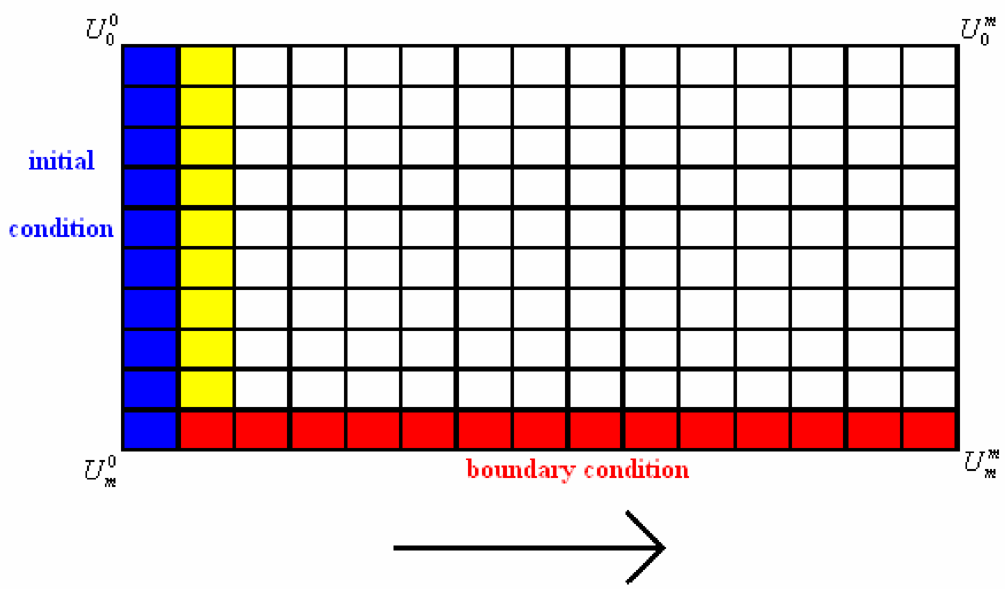
$$= \begin{bmatrix} (1 - 2\gamma d - ak)\ U_1^n + 2\gamma d\ U_1^n + kV_0^n \\ \gamma d\ U_0^n + (1 - 2\gamma d - ak)\ U_1^n + \gamma d\ U_2^n + kV_1^n \\ \vdots \\ \gamma d\ U_{m-2}^n + (1 - 2\gamma d - ak)\ U_{m-1}^n + \gamma d\ U_m^n + kV_{m-1}^n \\ \gamma d\ U_{m-1}^n + (1 - 2\gamma d - ak)\ U_m^n + \gamma d\ (U_{m+1}^{n+1} + U_{m+1}^n) + kV_m^n \end{bmatrix}$$

在這個題目當中，我們使用了 Crank-Nicolson method 的隱式算法。應此可以得知我們所需要的點如下圖(三)，而 stability 的條件也就沒什麼限制。



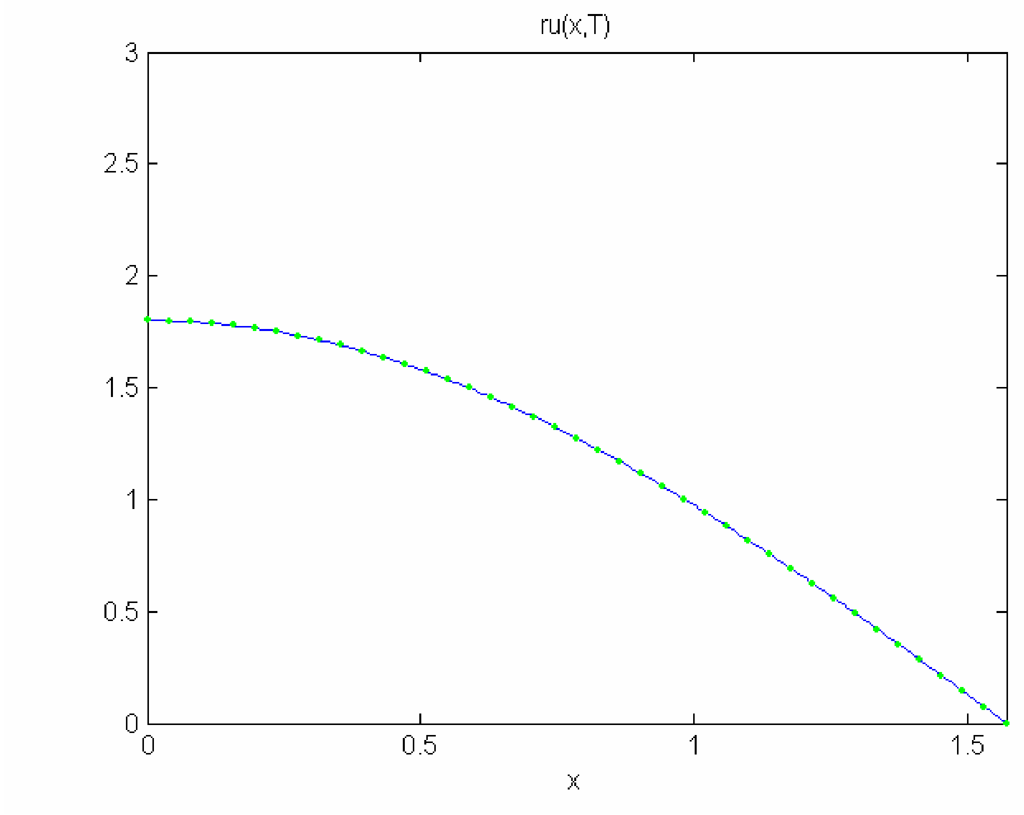
圖(三)

我們給 initial condition 及 boundary condition，可以得到的圖形為圖(四)，在這個圖形之中，我們給定紅色為 boundary condition，而且我們這次給的是 Newmann condition，因此只有下面 U_m 一段而已；藍色為 initial condition；黃色為我們一次所解出來的解。而我們所解順序為從時間 $t_1 \to t_n$，解出在同一時間下的 U^t。而 V 的狀況也是在類似的條件下，因此我們就不贅述了。

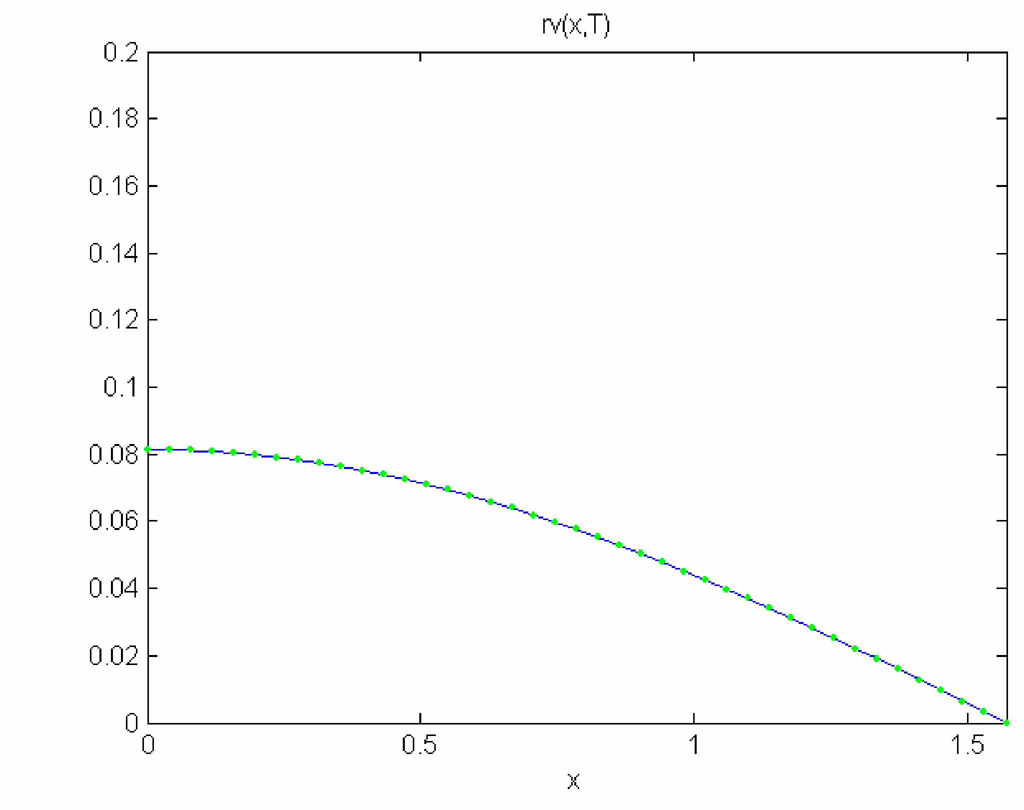


圖(四)

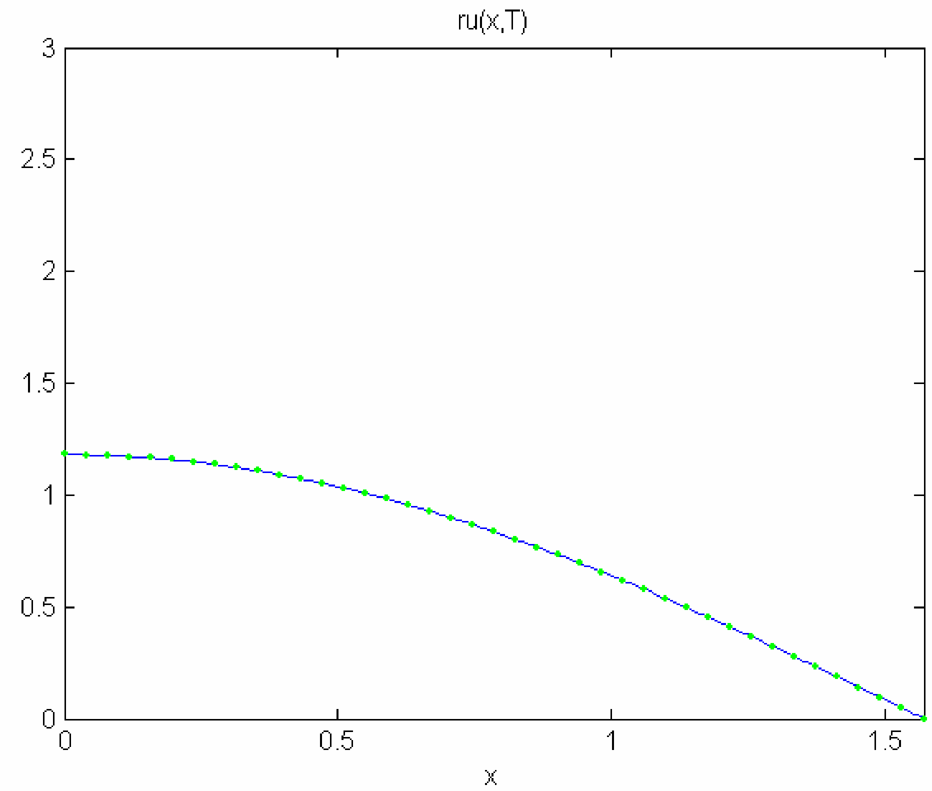
接下來我們以 $M = 39$ 來跑整個數值解，並且展示出 $t = 0.1, t = 0.5, t = 1$ 等這三個時間點的圖形。



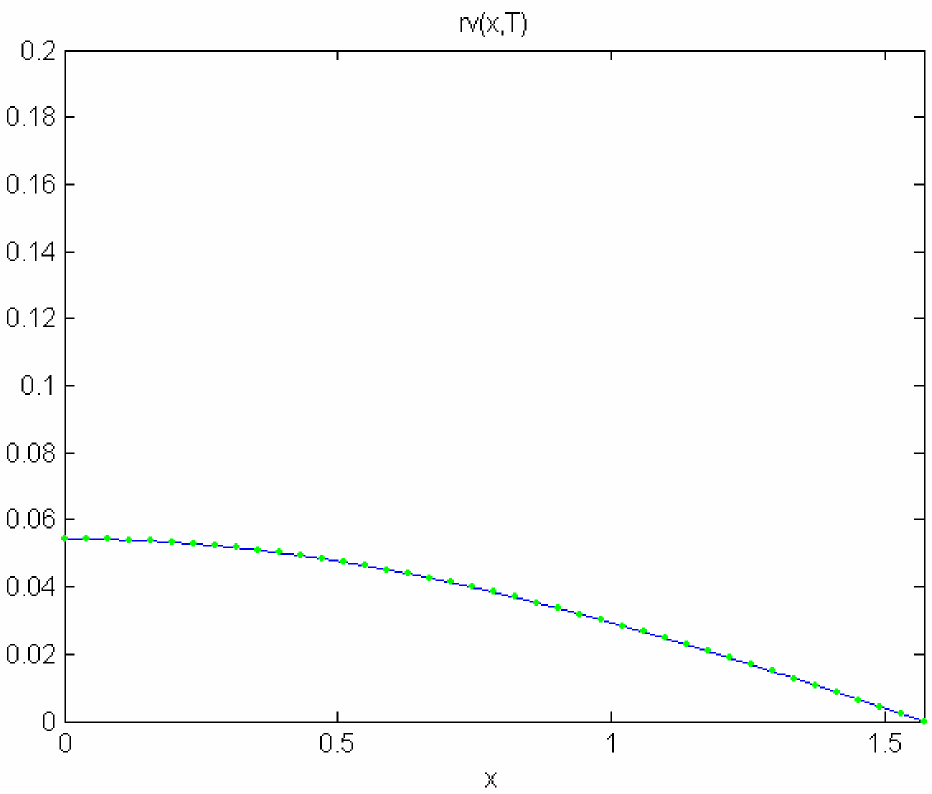
函數u在時間為$t = 0.1$的圖形



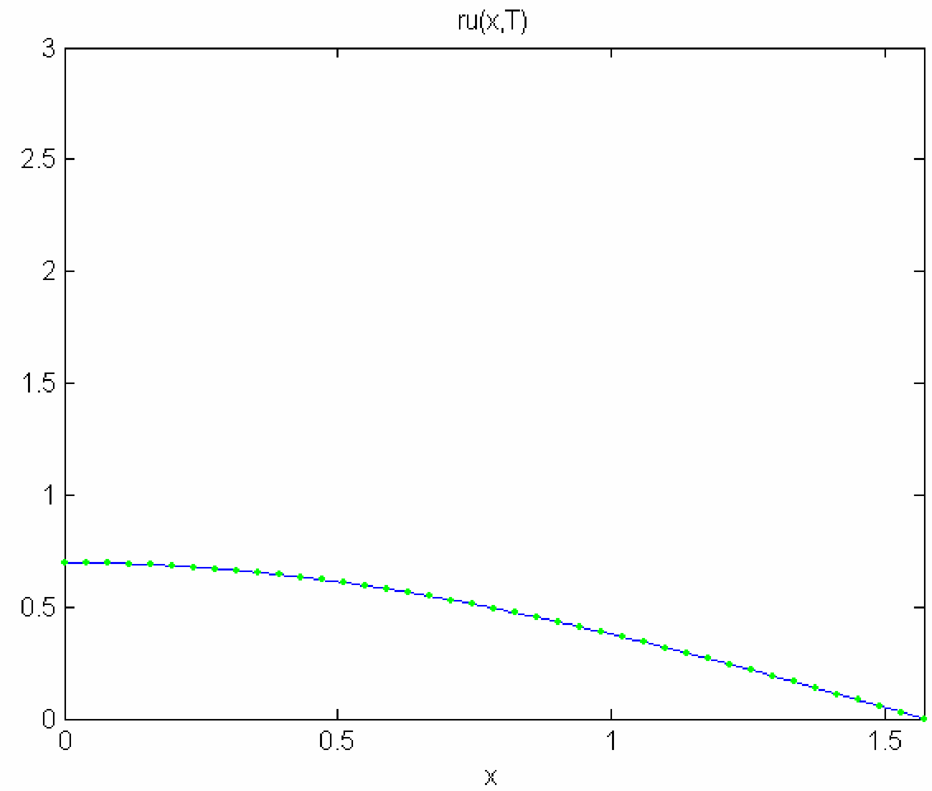
函數v在時間為$t = 0.1$的圖形



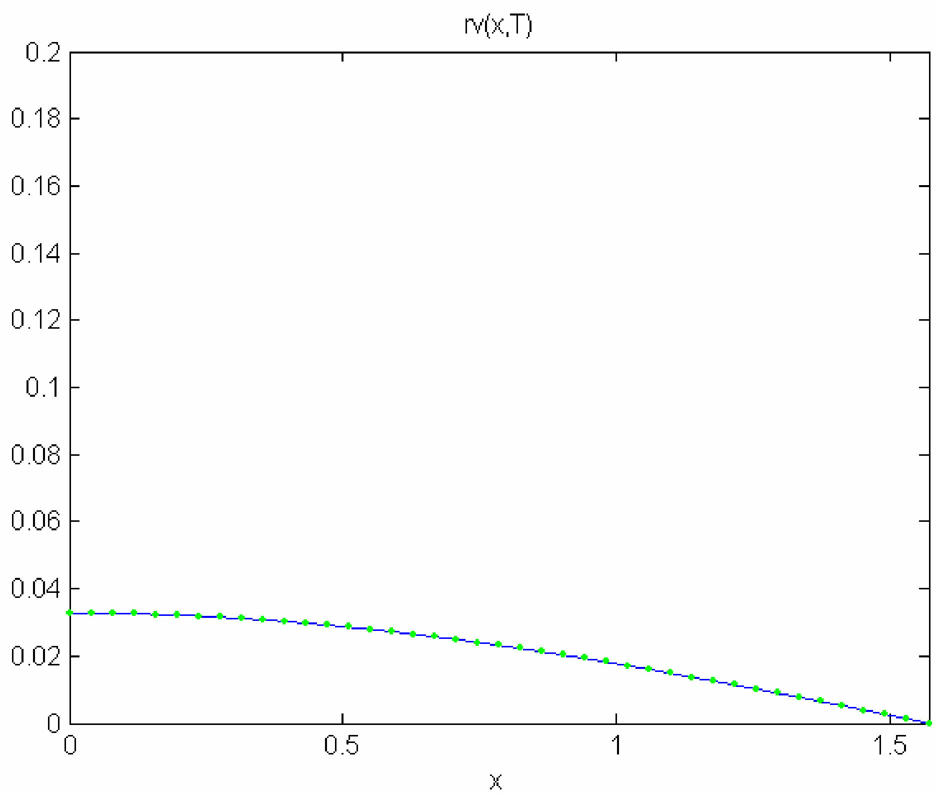
函數 u 在時間為 $t=0.5$ 的圖形



函數 v 在時間為 $t=0.5$ 的圖形



函數 u 在時間為 $t=1$ 的圖形



函數 v 在時間為 $t=1$ 的圖形

3. $\begin{cases} u_t = d_1\ u_{xx} + f(u,v) \\ v_t = d_2\ v_{xx} + g(u,v) \end{cases}$

$\begin{cases} u_x(0,t) = 0 \\ v_x(0,t) = 0 \end{cases}$ I.C. $\begin{cases} u(x,0) = \text{random number} \\ v(x,0) = \text{random number} \end{cases}$ B.C. $\begin{cases} u(\frac{\pi}{2},t) = 0 \\ v(\frac{\pi}{2},t) = 0 \end{cases}$

We take $x_i = ih,\ t_n = nk,$ where $h = \frac{1}{m+1},\ k = \frac{1}{n+1}$.

By Crank-Nicolson method,

$$\Rightarrow \begin{cases} \frac{U_i^{n+1} - U_i^n}{k} = \frac{d_1}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) + f(U_i^n, V_i^n) \\ \frac{V_i^{n+1} - V_i^n}{k} = \frac{d_2}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) + g(U_i^n, V_i^n) \end{cases}$$

$$\Rightarrow \begin{cases} U_i^{n+1} = U_i^n + \frac{d_1 k}{2h^2}\ (U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) + kf(U_i^n, V_i^n) \\ V_i^{n+1} = V_i^n + \frac{d_2 k}{2h^2}\ (V_{i-1}^n - 2V_i^n + V_{i+1}^n + V_{i-1}^{n+1} - 2V_i^{n+1} + V_{i+1}^{n+1}) + kg(U_i^n, V_i^n) \end{cases}$$

Let $\gamma = \frac{k}{2h^2}$.

$$\Rightarrow \begin{cases} -\gamma d_1\ U_{i-1}^{n+1} + (1+2\gamma d_1)\ U_i^{n+1} - \gamma d_1\ U_{i+1}^{n+1} = \gamma d_1\ U_{i-1}^n + (1-2\gamma d_1)\ U_i^n + \gamma d_1\ U_{i+1}^n + kf(U_i^n, V_i^n) \\ -\gamma d_2\ V_{i-1}^{n+1} + (1+2\gamma d_2)\ V_i^{n+1} - \gamma d_2\ V_{i+1}^{n+1} = \gamma d_2\ V_{i-1}^n + (1-2\gamma d_2)\ V_i^n + \gamma d_2\ V_{i+1}^n + kg(U_i^n, V_i^n) \end{cases}$$

$$\begin{bmatrix} 1+2\gamma d_2 & -2\gamma d_2 & 0 & & & 0 \\ -\gamma d_2 & 1+2\gamma d_2 & -\gamma d_2 & & & \\ & & & \ddots & & \\ & & & -\gamma d_2 & 1+2\gamma d_2 & -\gamma d_2 \\ 0 & & & 0 & -\gamma d_2 & 1+2\gamma d_2 \end{bmatrix} \begin{bmatrix} V_0^{n+1} \\ V_1^{n+1} \\ \vdots \\ V_{m-1}^{n+1} \\ V_m^{n+1} \end{bmatrix}$$

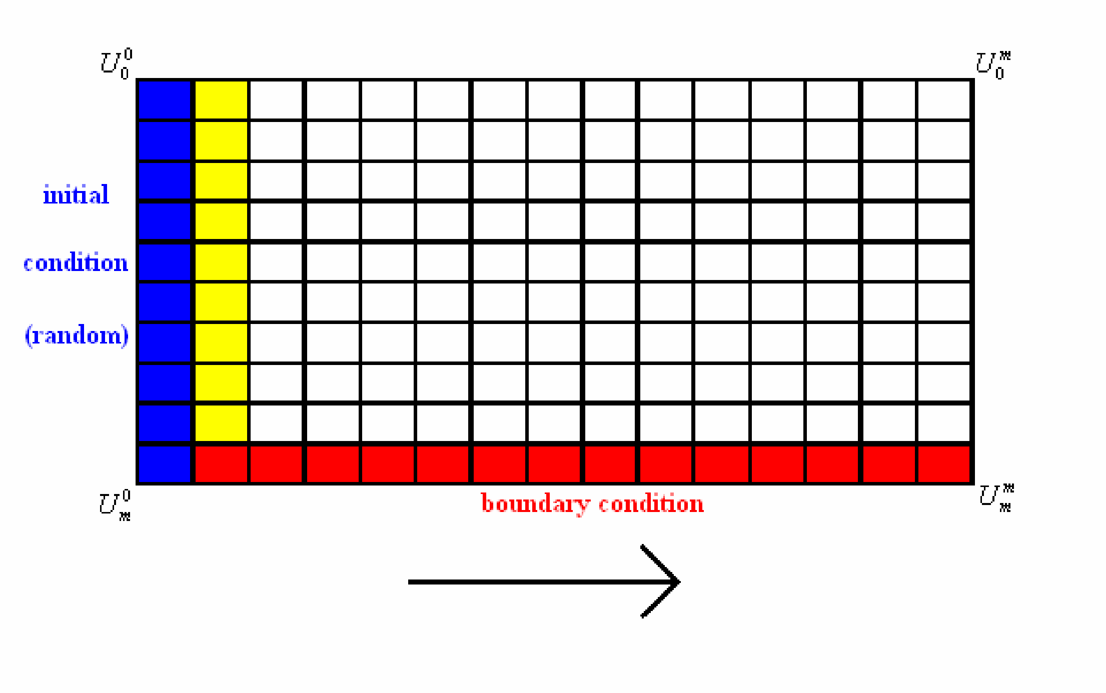
$$= \begin{bmatrix} (1-2\gamma d_2)\ V_1^n + 2\gamma d_2\ V_1^n + kg(U_i^n, V_i^n) \\ \gamma d_2\ V_0^n + (1-2\gamma d_2)\ V_1^n + \gamma d_2\ V_2^n + kg(U_i^n, V_i^n) \\ \vdots \\ \gamma d_2\ V_{m-2}^n + (1-2\gamma d_2)\ V_{m-1}^n + \gamma d_2\ V_m^n + kg(U_i^n, V_i^n) \\ \gamma d_2\ V_{m-1}^n + (1-2\gamma d_2)\ V_m^n + \gamma d_2\ (V_{m+1}^{n+1} + V_{m+1}^n) + kg(U_i^n, V_i^n) \end{bmatrix}$$

$$, \text{and} \begin{bmatrix} 1+2\gamma d_1 & -2\gamma d_1 & 0 & & & 0 \\ -\gamma d_1 & 1+2\gamma d_1 & -\gamma d_1 & & & \\ & & & \ddots & & \\ & & & -\gamma d_1 & 1+2\gamma d_1 & -\gamma d_1 \\ 0 & & & 0 & -\gamma d_1 & 1+2\gamma d_1 \end{bmatrix} \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \\ U_m^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} (1-2\gamma d_1)\ U_1^n + 2\gamma d_1\ U_1^n + kf(U_i^n, V_i^n) \\ \gamma d_1\ U_0^n + (1-2\gamma d_1)\ U_1^n + \gamma d_1\ U_2^n + kf(U_i^n, V_i^n) \\ \vdots \\ \gamma d_1\ U_{m-2}^n + (1-2\gamma d_1)\ U_{m-1}^n + \gamma d_1\ U_m^n + kf(U_i^n, V_i^n) \\ \gamma d_1\ U_{m-1}^n + (1-2\gamma d_1)\ U_m^n + \gamma d_1\ (U_{m+1}^{n+1} + U_{m+1}^n) + kf(U_i^n, V_i^n) \end{bmatrix}$$

這個 model 我們是從 Takashi Miura and Philip Maini 的文章所找到的，而在文章中只有提到說這是從生物數學這方面所得到的 model，但是並無詳細的說明出該 model 在模擬些什麼樣的問題。

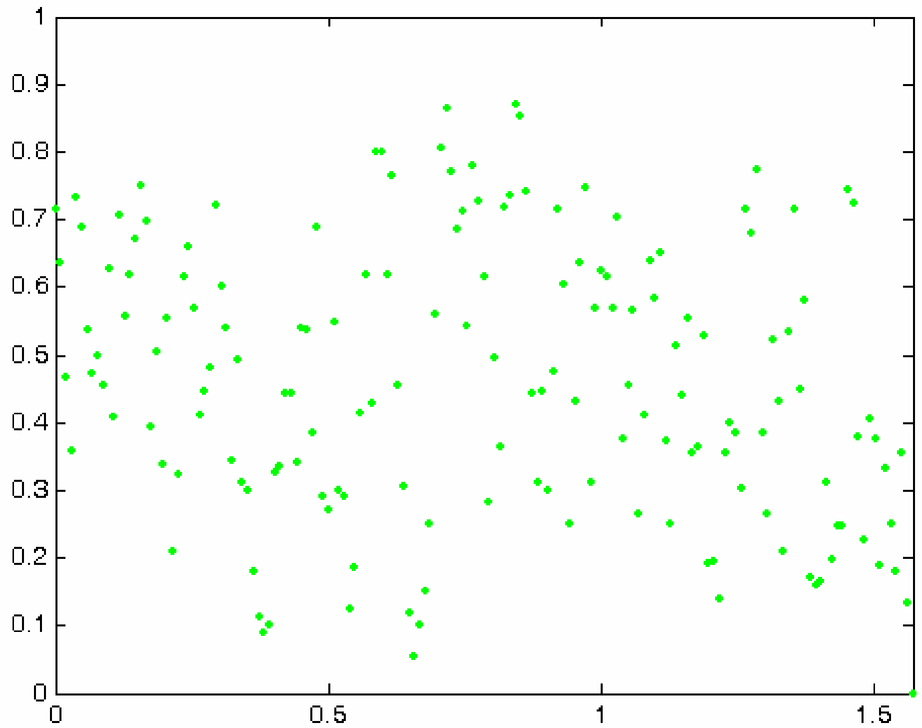
在這個題目當中，我們所使用的方法及給定的 boundary condition 都和上題相似，這次給的 boundary condition 一樣是 Newmann condition，因此只有 U_m 一段而已；藍色為 initial condition，而我們這次依 Takashi Miura and Philip Maini 的文章所給的 initial condition 為隨機給予的；黃色為我們一次所解出來的解。而我們所解順序為從時間 $t_1 \to t_n$，解出在同一時間下的 U^t。而 V 的狀況也是在類似的條件下，因此我們就不贅述了。



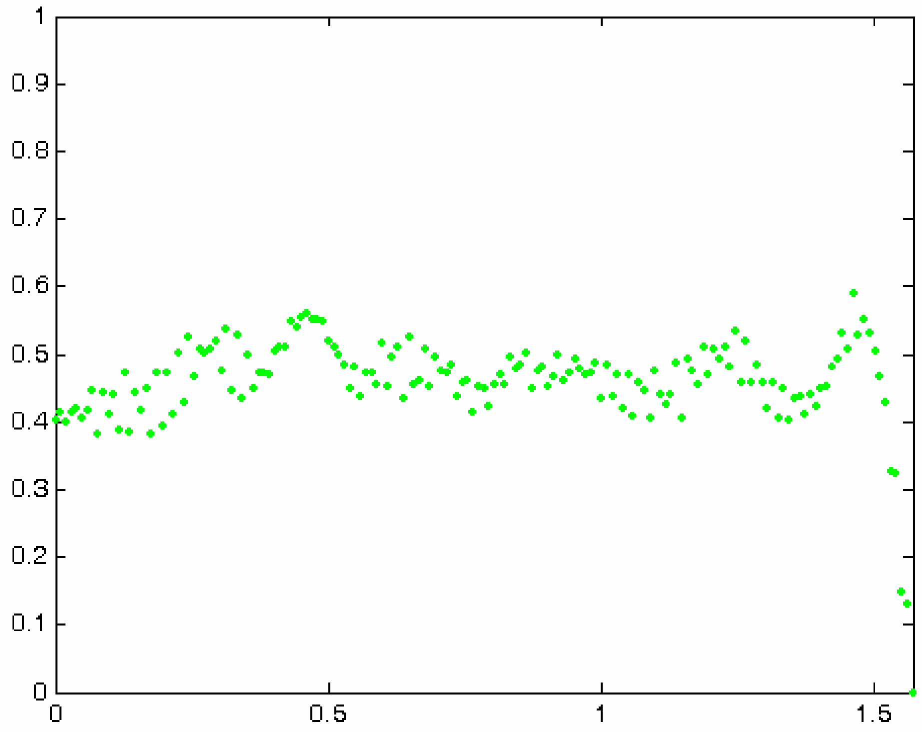
圖(五)

接下來我們取 $M = 159$ 以及時間 $t = 0.25, t = 0.5, t = 1, t = 2, t = 3, t = 4$ 等，不同的時間點來看整個數值解的圖形。

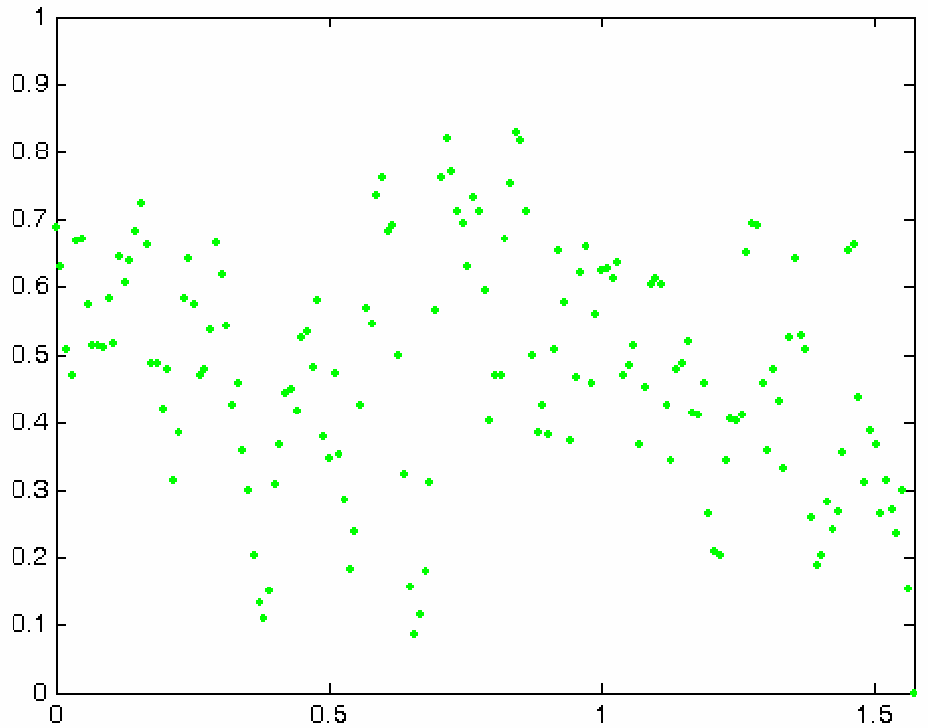
而這個圖形不管我們初始值取任何的 random number，最後都會趨向一個穩定態。



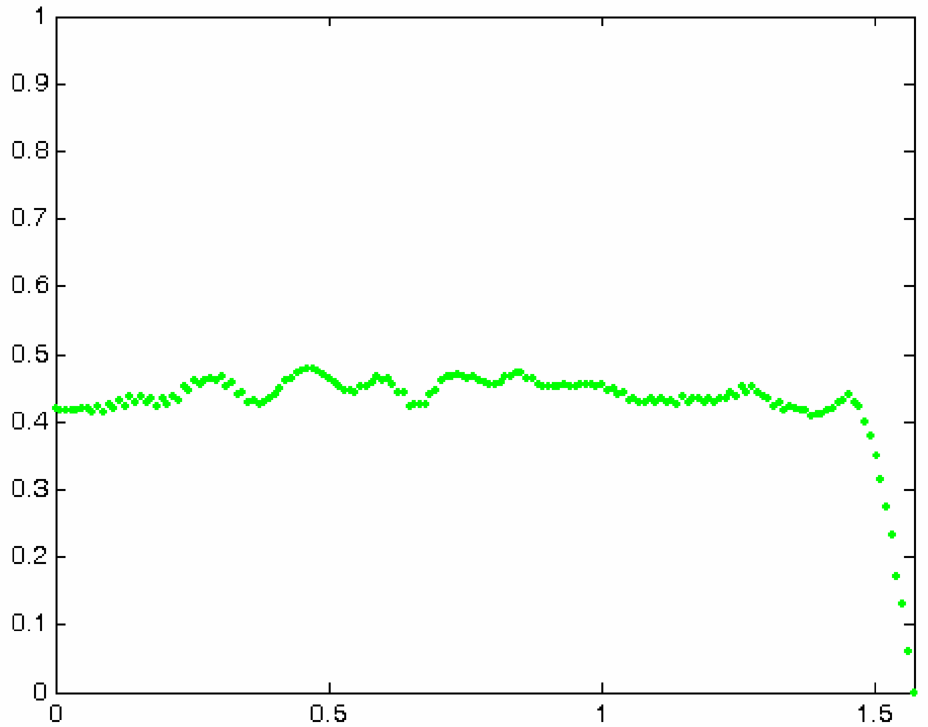
函數 u **在** $M = 159$ **及** $t = 0.25$ **的圖形**



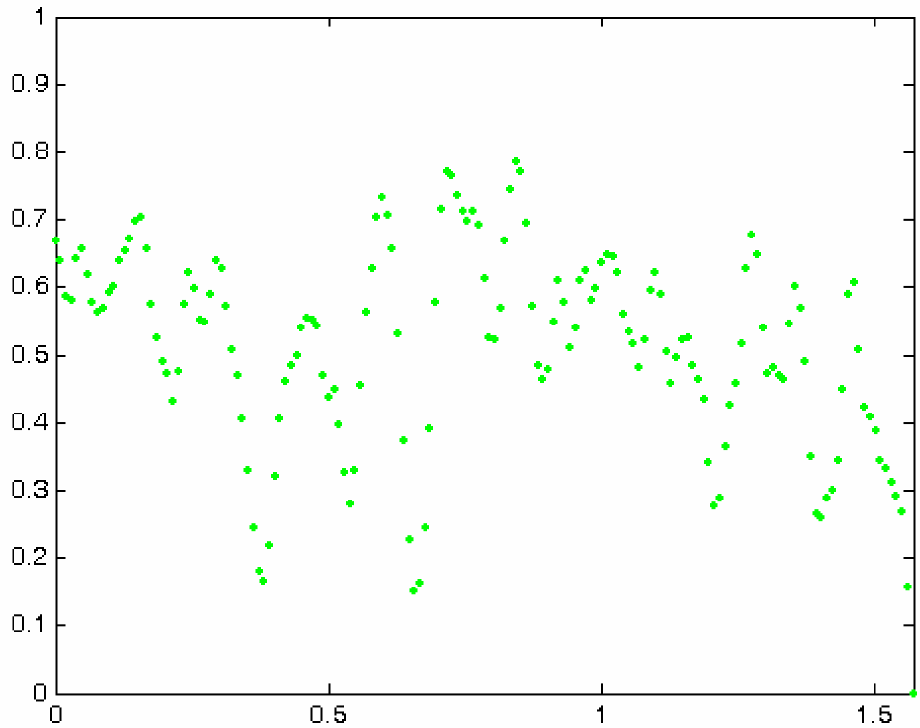
函數 v **在** $M = 159$ **及** $t = 0.25$ **的圖形**



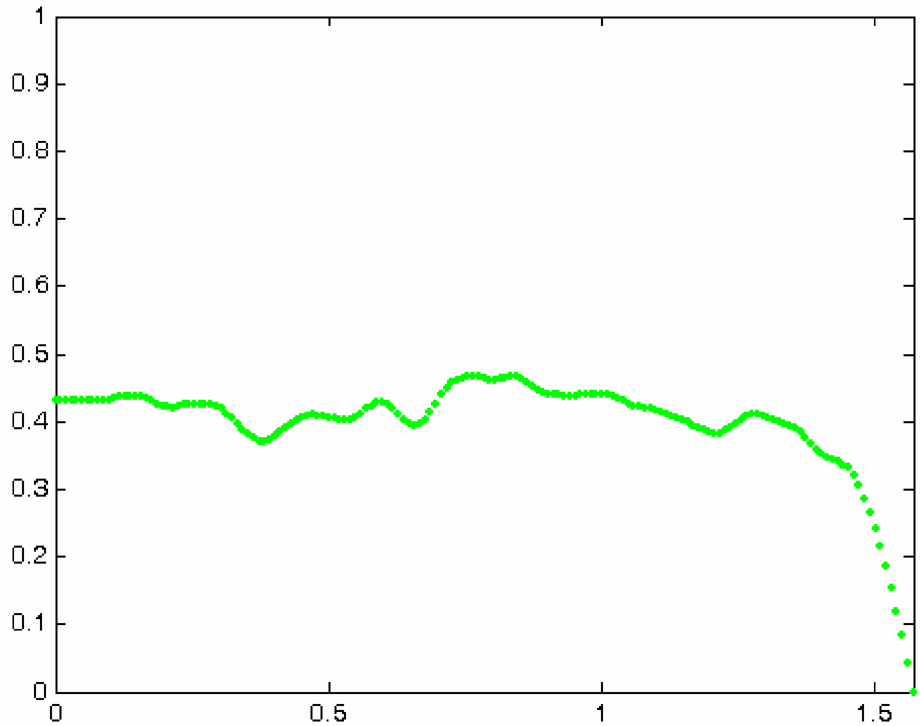
函數 u **在** $M=159$ **及** $t=0.5$ **的圖形**



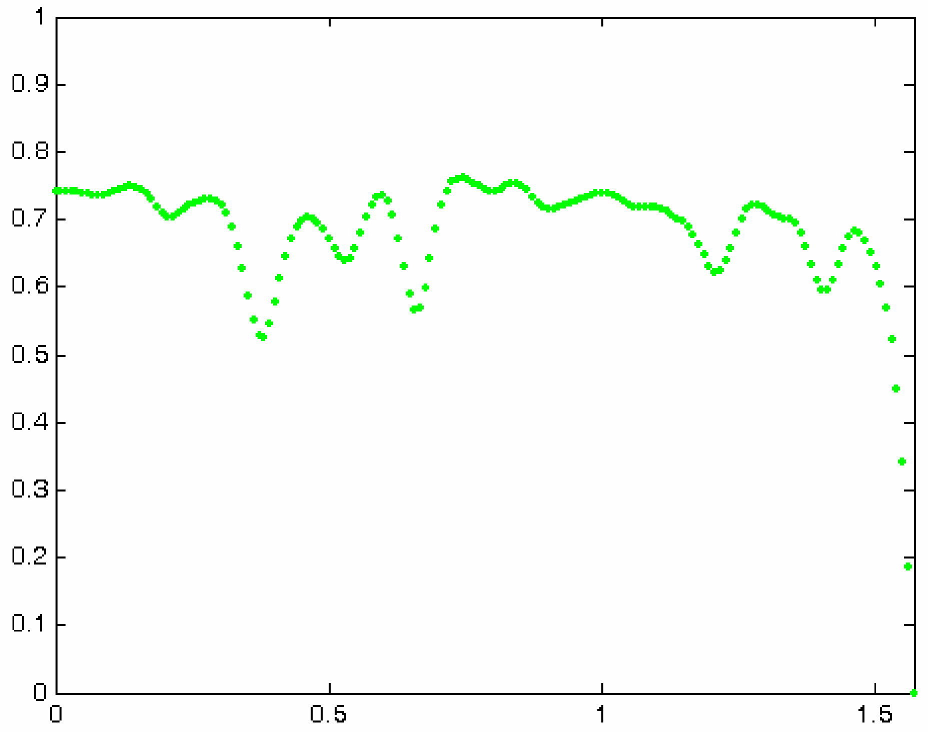
函數 v **在** $M=159$ **及** $t=0.5$ **的圖形**



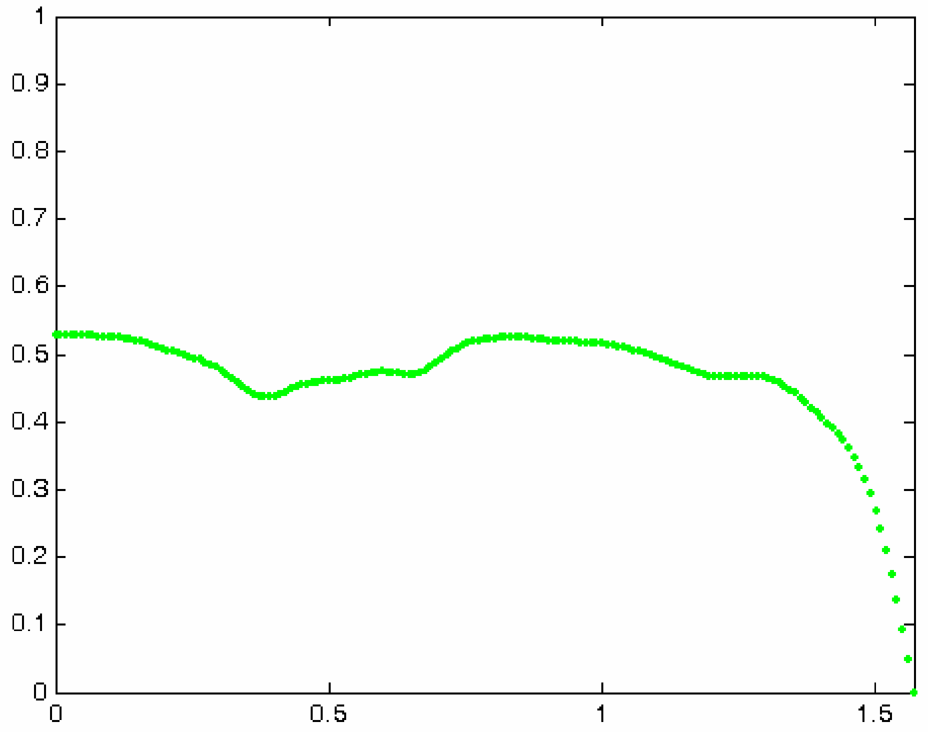
函數u在$M=159$及$t=1$的圖形



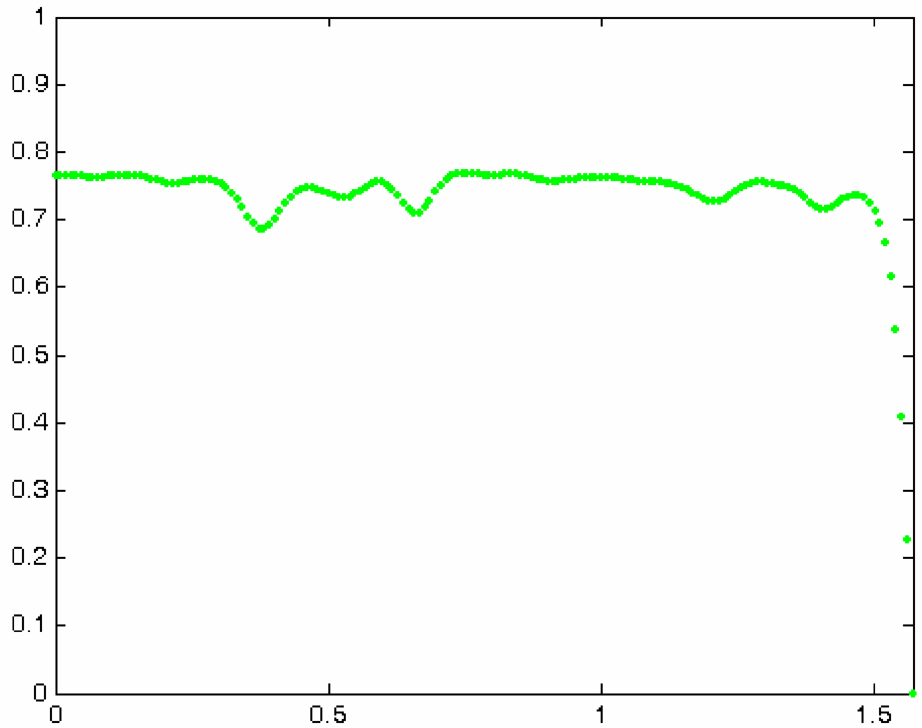
函數v在$M=159$及$t=1$的圖形



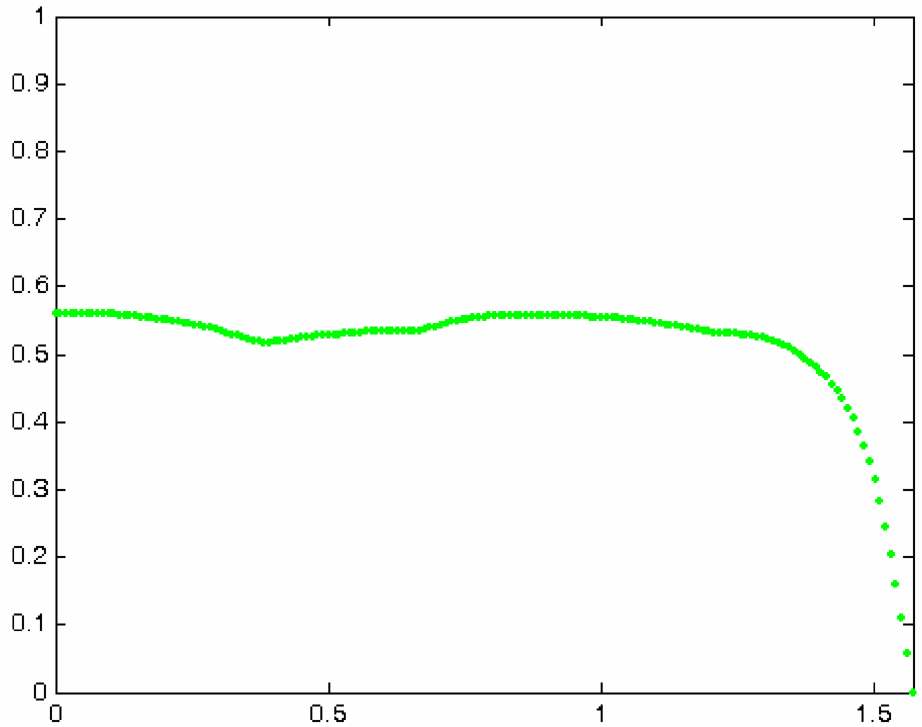
函數u在$M=159$及$t=2$的圖形



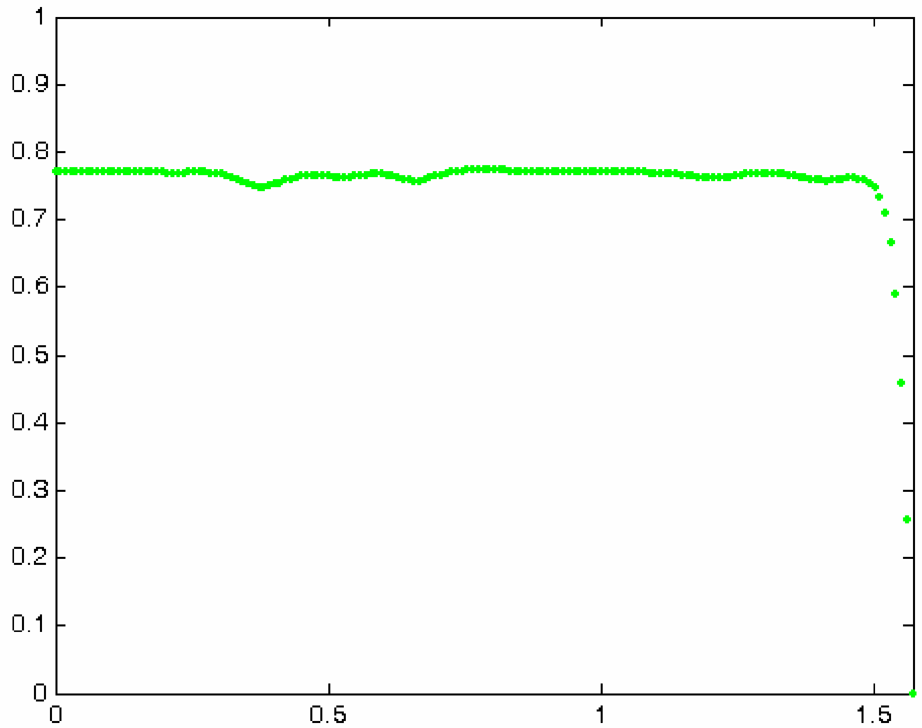
函數v在$M=159$及$t=2$的圖形



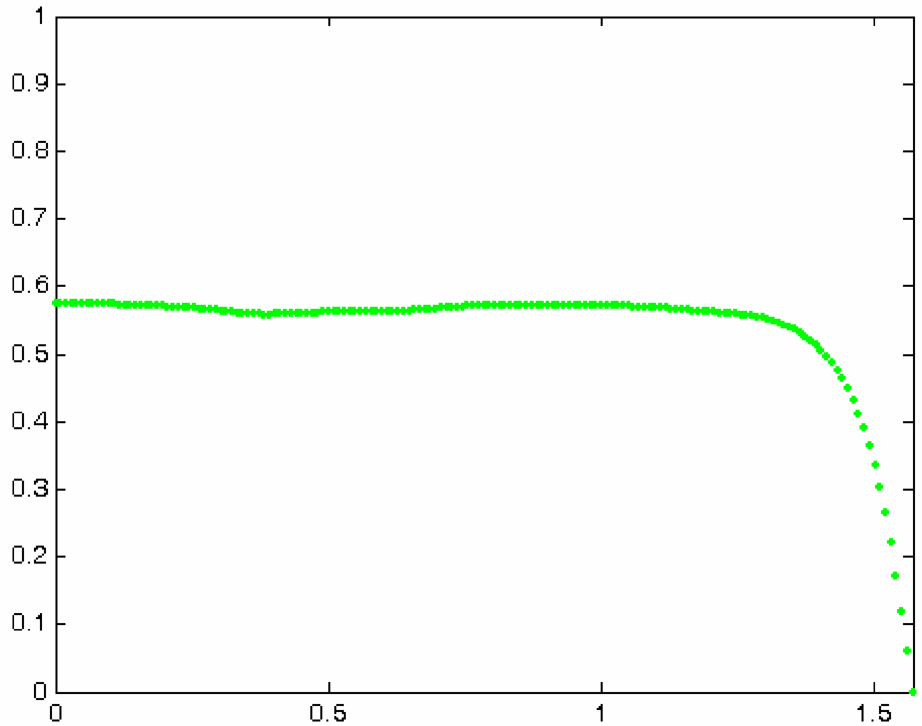
函數u在$M=159$及$t=3$的圖形



函數v在$M=159$及$t=3$的圖形



函數u在$M=159$及$t=4$的圖形



函數v在$M=159$及$t=4$的圖形